On a Kreigspiel Problem of Lloyd Shapley.

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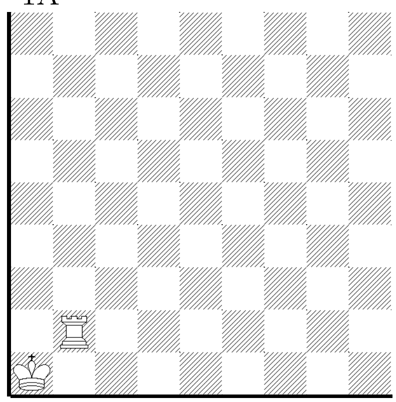
1. Statement of the Problem. In Lloyd Shapley's unpublished book of kriegspiel problems, "The Invisible Chessboard", there is a problem, dated "c. 1960", of mate with white king and rook against black king on a quarter infinite board. We denote the squares of the board by the positions (x, y), with integers $x \geq 1$ and $y \geq 1$ representing the column and row respectively, and with the position (1,1) referred to as the southwest corner. The initial position has the white king in the southwest corner and the rook at (2,2):

1A



The position of the black king is completely unknown, and the problem is for white to checkmate black king with probability one no matter where he is and no matter what he does. (The book of Shapley has the equivalent problem with the positions of the white king and rook interchanged.)

The fact that rook and king can give mate against king in kriegspiel on a standard 8 by 8 board has been known for a long time. One strategy for achieving the mate is given in Boyce (1981). In Ciancarini and Favini (2009), perfect play procedures have been computed for all starting positions, the longest forced mate being 37 moves long.

A chess problem, due to Simon Norton, of mate with king and rook vs. king on a quarter infinite board, with white king and rook as in position 1A and black king at (3,3), is mentioned in Berlekamp, Conway and Guy (1982). The problem is to find the minimum number of moves to mate, and its solution has relevance here.

2. Solution of the Problem. The solution proceeds in three stages. Stage 1: Safely getting one horizontal and one vertical check in order to obtain an upper bound on the position of the black king. Stage 2: Bounding the black king inside a rectangle with the white rook at the northeast corner guarded by the white king. Stage 3: Bringing the black king back to the edge and mating in the corner.

Below, a method of carrying out these three stages successfully is outlined. The most difficult stage is the last as will be seen. It is interesting to note that on a board unbounded in only one direction, say the board (x, y) with $1 \leq x < \infty$ and $1 \leq y \leq \mathrm{N}$ for some finite N, it seems impossible to mate with probability one. This is because the method of stage 2 cannot be carried out. The method is not claimed to be efficient. In fact, it can easily be improved by a move or two in several places. The objective is to show that mate of the black king can be accomplished.

Stage 1. The first four moves in setting up the desired checks are

$$\text{R}\to(1,2),\ \text{R}\to(2,2),\ \text{K}\to(1,2),\ \text{and}\ \text{R}\to(2,1).$$

The first two moves are to make sure that the black king is not in the first or second columns. If either move is a check, we have our vertical check. If not, the other two moves set up the main search for a vertical check. The next five moves are

$$\text{R}\to(2,2),\ \text{R}\to(2,3),\ \text{R}\to(2,1),\ \text{R}\to(k,1),\ \text{and}\ \text{R}\to(2,1),$$

where k is some integer greater than 2. The first two moves are to make sure that the black king is not on the first, second or third rows. Then the white rook has time to try for a check on column k safely before returning to the protection of the king. These five moves are repeated indefinitely until a check occurs.

With the proper choice of the numbers, k, white can achieve probability one of eventually checking the black king, no matter where it starts or how it moves. One method of achieving this is as follows. On the nth trial, independently choose k at random equally likely among the numbers $\{3,4,5,\ldots,10n\}$. Eventually, say for n greater than some n^*, $10n$ will be larger than the column containing the black king. For $n>n^*$, the probability of choosing the column of the black king is $1/(10n-2)$. Since the sum of these numbers is infinite, the Borel-Cantelli Lemma implies that the probability is one that the black king will eventually be checked.

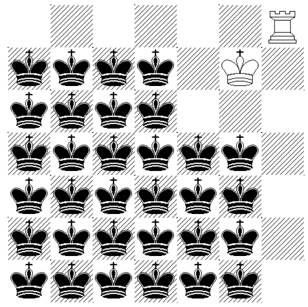
After one check is obtained, the corresponding check in the other direction may be found in a similar manner. For example, if a vertical check found the black king to be in column k, white could send his rook out to column $10^{10}k$, to start a more efficient search for the row check using the same probabilistic procedure. If necessary, the rook may be sent out even further. With probability one, white will eventually obtain a row check and have knowledge of a rectangle in which the black king is known to be.

Stage 2. We use a method related to the solution of Simon Norton's problem. Suppose the black king is known to be in the rectangle with northeast corner, (m,n) and suppose the white king is in the southwest corner. We now indicate how white can bound the black king into a larger rectangle, with the rook at $(m+n+5,m+n+5)$ guarded by the white king. Assume wlog that $m\geq n$. First, white sends the rook out to row $n+2$, and then out to $(n+2,m+n+5)$. After this, the black king must be in the rectangle with northeast corner $(m+1,n+2)$. Then white sends the king out the diagonal to $(n+2,n+2)$. If black tries to interfere with this, then white gets new information and can bound black into an even smaller rectangle. After this, the black king must be in the rectangle with northeast corner $(2n+3,m+n+4)$. So white moves his rook out to $(m+n+5,m+n+5)$ and continues to move his king out the diagonal to $(m+n+4,m+n+4)$. Black cannot interfere with this movement.

Stage 3. After the black king has been confined to a rectangle below the rook with the white king protecting the rook, the black king may be (slowly) brought back to the southwest corner. We say that white gains one every time he has reduced the size of the confining rectangle by one row or one column, as indicated by the rook. The method seems to require randomization. In fact, we give a strategy that works in stages to obtain a positive expected gain per stage. Our method allows the black king to escape the rectangle with a small probability and requires us to reconfine him in a necessarily larger rectangle. But since the expected gain is positive, we will with probability one be able to bring the black king to the corner for mate.

First the white king should be situated inside the confining rectangle as in positiion 2A.

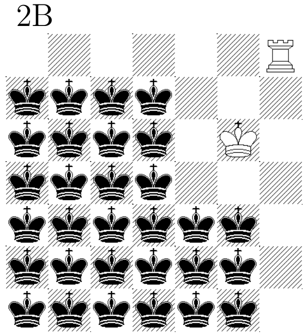
2A



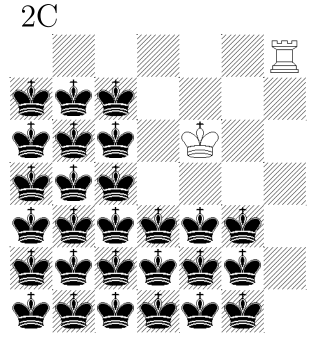
If the white king is outside the confining rectangle, this may be achieved by moving the rook back two squares.

Once inside the rectangle, the white king moves through position 2B to achieve the critical position, 2C. It is easy to obtain 2B from 2A; just try to move directly to that position, or if that is not possible then move the king one square to the left achieves a position which is a reflection of 2B.

2B



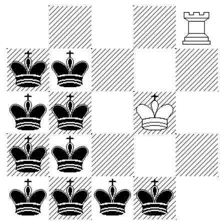
2C



We use a move description similar to that found in Ferguson (1992), except that here a relative move notation is used, with ‘l’, ‘r’, ‘u’, and ‘d’ denoting left, right, up and down respectively. Thus Kr indicates that the king tries to move right. Similarly, Kul indicates the king tries to move diagonally up and left. Moves of the rook more than one square are indicated by repeated letters; thus, Rrrr indicates that the rook moves right three squares. If an attempted move by the king is successful, play procedes to the right across the page. If an attempted move receives a ‘no’ from the referee, the next move tried falls below it in the same column, sometimes several rows below. Thus in 2B below, white tries king to the left. If that is a move, we are in position 2C; if that is not a move then white moves the rook to the left one square, and tries to move the king up left, etc. We say white gains 1 with the moves Rl or Rd, and loses 1 with Rr or Ru.

We now show that it is possible to go from 2B to 2C without loss. In fact, white will gain at least one unless black prevents the white king’s first move to the left in 2B. Here are the moves.

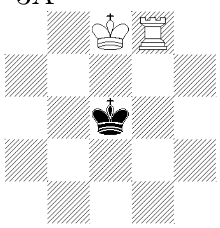
2B



Kl	2C			
Rl	Kul	2A	gains 1	
	Ku	Kl	2A	
		Kul	Kd	2A
			3A	
		3A (reflected)		

We show that from position 3A we may always achieve position 2B or position 3A again, moved one step to the right

3A



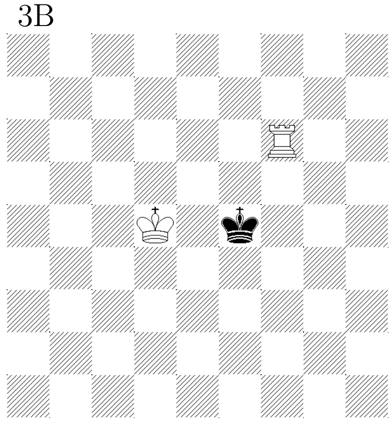
Kl	Kd	2B		
	Rl	Kdl	2B	gains 1
		Kdr	Kl	2A
			3A (reflected)	

At the critical position 2C, white mixes the two simple strategies of moving the king Kd-Ku or Kl-Kr with probability $(1-\epsilon)/2$ each and two complex strategies that begin by moving the rook left or down with probability $\epsilon/2$ each. The strategy that begins with Rl is

Rl	2B				
if+	Rrr	Kr	Rd	2A	
		Kur	Kd	Rl	2A
			Rl	2A	
		3B (failure)			

The position defining failure is

3B



The strategy continues for failure with

Ku	Rrrr	Kur	Kur	Ruuu	Kur	Kr	2B		
						Kur	Kd	2B	
							Kr	Kd	2A
								3A	

for a loss of seven in the longest variation. The strategy beginning with Rd is defined symmetrically.

Let us check that this strategy gives a positive expected gain no matter where the black king is situated within the rectangle, 2C.

Case 1. The black king is not in the rightmost column or the highest row. If the white king moves there is never any loss; if the rook moves there is a gain of 1. The expected gain is at least ϵ.

Case 2. The black king is in the rightmost column but not at the top of that column (or symmetrically, in the highest row but not on the right of that row). Failure cannot occur in this case, so the expected gain is at least $\epsilon/2$, achieved when the rook moves down.

Case 3. The black king is at the top of the rightmost row (symmetrically at the right of the highest row). If the king moves down or the rook moves down, there is a gain of 1. Otherwise the loss is at most seven if the rook moves left. The expected gain is $(1-\epsilon)/2+\epsilon/2-7\epsilon/2=(1/2)-(7/2)\epsilon$. This is positive if $\epsilon<1/7$.

In summary, by choosing $0<\epsilon<1/7$, we obtain a positive expected gain no matter where the black king is situated in the confining rectangle.

Once the black king is brought to one of the edges of the board, it is easy to perform the checkmate using methods suggested by Boyce (1981).

References.

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