**Gambler's ruin in three dimensions**

Three gamblers with initial fortunes, a, b, and c, play a sequence of fair games until one of them is ruined. It is desired to find the probability that one of them, say gambler 3, is the one that is ruined. The sum of the fortunes $a + b + c = s$ remains fixed throughout the game. We model this as a Brownian motion in the plane of the equilateral triangle with barycentric coordinates $(x, y, z)$ starting at the initial point, $(a, b, c)$, and seek the probability that the Brownian motion first exits the triangle along the edge $z = 0$.

The method of solution used here is to find a conformal mapping of the equilateral triangle onto the unit circle that maps the point $(a, b, c)$ into the origin. Then the desired probability will be the proportion of the image of the edge $z = 0$ on the circumference of the circle.

We take the equilateral triangle, $\Delta$, in the plane to be the triangle with vertices $(-1, 0)$, $(1, 0)$, and $(0, \sqrt{3})$. The mapping of $\Delta$ into the unit circle is effected in two parts; first, a map of the triangle into the upper half plane, and then a map of the upper half plane into the circle. We take the mapping of the triangle into the upper half plane to be the inverse map of

$$w = \frac{2}{B(1/2, 1/3)} \int_0^z \frac{dt}{(1 - t^2)^{2/3}} \tag{1}$$

where $B(\alpha, \beta)$ is the beta function, and $B(1/2, 1/3) = 2 \int_0^1 (1 - t^2)^{-2/3} dt$. This maps the upper half plane into $\Delta$, mapping $z = 0$ into $w = 0$, $z = 1$ into $w = 1$, $z = -1$ into $w = -1$, and $z = \infty$ into $w = i\sqrt{3}$. This is an example of the Schwarz-Christoffel transformation.

After mapping $w$ into $z$ by the inverse of this transformation, we map the upper half plane into the unit disc by means of a Möbius transformation that maps an arbitrary point $z_0 = x_0 + iy_0$ with $y_0 > 0$ of the upper half plane into the origin. This is

$$t = \frac{z - z_0}{i(z - \overline{z}_0)}. \tag{2}$$

This maps an arbitrary point $z = x_1$ of the real axis into the point

$$t = \frac{2y_0(x_0 - x_1) + i[y_0^2 - (x_0 - x_1)^2]}{y_0^2 + (x_0 - x_1)^2} \tag{3}$$

on the unit circle. In particular, $z = x_0$ goes into $t = i$, $z = x_0 - y_0$ goes into $t = 1$, and $z = x_0 + y_0$ goes into $t = -1$. The main use of (3) is to find the Args of the images of $z = -1$ and $z = +1$. These are

$$\begin{aligned} \theta_{-1} &= \arctan \frac{y_0^2 - (x_0 + 1)^2}{2y_0(x_0 + 1)} \\ \theta_1 &= \arctan \frac{y_0^2 - (x_0 - 1)^2}{2y_0(x_0 - 1)} \end{aligned} \tag{4}$$

respectively. The desired probability is $(\theta_1 - \theta_{-1})/2\pi$ where $z_0$ is the image of $(a, b, c)$ under the inverse map (1).

The only difficulty in computation is the map (1). First consider the case of $z = iy$ pure imaginary. We find

$$\begin{aligned} w &= \frac{2}{B(1/2,1/3)} \int_0^{iy} \frac{dt}{(1-t^2)^{2/3}} = \frac{2i}{B(1/2,1/3)} \int_0^{y} \frac{dx}{(1+x^2)^{2/3}} \\ &= iB(1/2, 1/6, \frac{y^2}{1+y^2})/B(1/2,1/3) \end{aligned} \tag{5}$$

Here, $B(\alpha, \beta, z) = \int_0^z x^{\alpha-1}(1-x)^{\beta-1} dx$ is the incomplete beta function. Since $w$ must converge to the top point of $\Delta$ as $y$ goes to $\infty$, we must have $B(1/2, 1/6) = \sqrt{3}B(1/2, 1/3)$. The mapping (5) becomes

$$w = i\sqrt{3}B(1/2, 1/6, \frac{y^2}{1+y^2})/B(1/2, 1/6)$$

which is $i\sqrt{3}$ times the beta distribution function with parameters 1/2 and 1/6 evaluated at $y^2/(1+y^2)$.

As an example, let us compute the probability that Player 3 is ruined when $a = b = .5$ and $c = 1$. This point in barycentric coordinates corresponds to the point $w = i\sqrt{3}/2$ in $\Delta$. The median of the beta distribution with parameters 1/2 and 1/6 is .9510, so we may solve $y^2/(1+y^2) = .9510$ to find $y_0 = 4.404$ and $x_0 = 0$. Substituting into (4), we find that $\theta_{-1} = \arctan((y_0^2 - 1)/(2y_0)) = 1.1243$. Since $\theta_1$ is placed symmetrically across the $y$-axis, we have that the probability Player 3 is ruined first is $2((\pi/2) - 1.1243)/(2\pi) = .1421$.

## Gambler's Ruin on the sphere

Here is another version of the problem. Let $(X_t, Y_t, Z_t)$ be Brownian motion in the upper octant of 3-space, starting at a point $(X_0, Y_0, Z_0) = (x, y, z)$ with $x > 0$, $y > 0$ and $z > 0$. The Brownian motion stops the first time the motion exits the upper octant. We first solve the problem of finding the probability that the motion exits the upper octant on the $x$-$y$ plane, that is, on the plane $z = 0$. This analysis extends to any number of players. Unfortunately, it is not the real gambler's ruin problem.

Let $T_x$ denote the time that a one-dimensional Brownian motion $X_t$ first hits $x > 0$ starting at $X_0 = 0$. By the reflection principle,

$$\begin{aligned} \mathrm{P}(T_x < t) = 2\mathrm{P}(X_t > x) = 2\mathrm{P}(X_t\sqrt{t} < \frac{x}{\sqrt{t}}) &= 2\int_{x/\sqrt{t}}^{\infty} \frac{1}{2\pi} e^{-u^2/2}\, du \\ &= \frac{1}{\sqrt{\pi}} \int_{x^2/(2t)}^{\infty} e^{-v} v^{-1/2}\, dv = 1 - I(\frac{1}{2}, \frac{x^2}{2t}) \end{aligned}$$

where $I(\alpha, x)$ represents the gamma distribution function. From this we may find the density

$$f_{T_x}(t) = \frac{x}{\sqrt{2\pi}} e^{-x^2/(2t)} t^{-3/2}$$

for $t > 0$. Using this, we may find the probability that the upper octant is exited on the $x$-$y$ plane as $\mathrm{P}(T_z < T_x, T_z < T_y)$ where $T_x$, $T_y$ and $T_z$ are independent. We find

$$\begin{aligned}
\mathrm{P}(T_z < T_x, T_z < T_y) &= \int_0^\infty \mathrm{P}(T_x > t)\mathrm{P}(T_y > t) f_{T_z(t)}\, dt \\
&= \int_0^\infty I(\frac{1}{2}, \frac{x^2}{2t}) I(\frac{1}{2}, \frac{y^2}{2t}) \frac{z}{\sqrt{2\pi}} e^{-z^2/(2t)} t^{-3/2}\, dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty I(\frac{1}{2}, \frac{x^2u^2}{z^2}) I(\frac{1}{2}, \frac{y^2u^2}{z^2}) e^{-u^2}\, du.
\end{aligned}$$

This is related to a gambler's ruin problem on the sphere. The projection of the Brownian motion $(X_t, Y_t, Z_t)$ onto the sphere $x^2 + y^2 + z^2 = 1$ is a Brownian motion on the sphere. Therefore the solution of the above problem is the solution to the problem of Brownian motion on the sphere, starting at $(x, y, z)$ on the upper octant of the sphere, of finding the probability of exiting the upper octant at $z = 0$. In $d$-dimensions, $X_t$ is the "skew-product" of a Brownian motion on the sphere, $\Theta_t$, times a "Bessel process", $R(t)$, in the sense that there is a scale-change in the process on the sphere, i.e. $X_t = R(t) \cdot \Theta_{\int dt/R(t)}$. See Ito and McKean, Chapter 6.

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