NIM, TRIM and RIM

James A. Flanigan

University of California, Los Angeles, 1980

The solutions of certain combinatorial games are of a particularly nice form. For the games we shall discuss, the detection of winning strategies utilizes the representation of integers in certain bases, as well as certain kinds of addition.

Take-away games are played with several piles of chips. Two players alternate moves. A move consists of selecting some of the piles, and then taking chips from each of the selected piles. The rules of a particular game specify how many piles may be selected and how many chips may be taken. A player loses if he is unable to move on his turn.

The most celebrated take-away game is **Nim**, which was analyzed by C. L. Bouton [1] in 1902. In Nim, a player selects just one pile and removes from it any positive number of chips. For a particular game (also called a position), the **base 2 game array** is formed by writing each pile size in binary notation. If the piles have sizes 3, 4, 9, and 11, for example, the array is

$$\begin{aligned} 3 &= 0\ \ 0\ \ 1\ \ 1 \\ 4 &= 0\ \ 1\ \ 0\ \ 0 \\ 9 &= 1\ \ 0\ \ 0\ \ 1 \\ 11 &= 1\ \ 0\ \ 1\ \ 1. \end{aligned}$$

In Nim, the second player to move can force a win if and only if for each column in the base 2 game array, the sum modulo 2 of the entries is zero. In the example, taking 3 chips from the pile of size 4 is the unique winning move for the first player.

In 1910, E. H. Moore [2] analyzed a generalization of Nim known as Nim_k. For fixed $k \geq 2$, to move in Nim_k select from 1 to $k-1$ of the piles. From each of the selected piles, take any positive number of chips. Nim_2 is just Nim. In Nim_k, the second player to move can force a win if and only if for each column in the base 2 game array, the sum modulo k of the entries is zero. For example in Nim_3 with piles of sizes 3, 4, 9, and 11, the first player has a unique winning move: Take 2 chips from the pile of size 9 and 4 chips from the pile of size 11.

The simple, but rather unexpected analysis of Nim and Nim_k leads naturally to a certain **problem**. Find a game whose solution uses notation in base 3 and addition modulo 3. Or, more generally, such a game when 3 is replaced by any integer $k \geq 3$. No such game seems to be in the literature. Let us introduce some games we have devised.

Trim is a take-away game in which a player selects a single pile and from it takes any positive number of chips, T_1 say. Furthermore, he has the option of taking some more chips from a second pile, which may be the same pile from which the T_1 chips were taken. The number of chips, T_2 say, taken from the second pile must be a power of 3 and must satisfy $2T_2 > T_1$. These restrictions distinguish Trim from Nim_3.

As its name suggests, (Tri)m is solved by first expressing each pile size in base 3 notation. The second player to move can force a win if and only if for each column in the **base 3 game array**, the sum modulo 3 of the entries is zero. As an example, consider the game of Trim beginning with piles of sizes 3, 5, 11, and 16.

$$\begin{aligned} 3 &= 0\ \ 1\ \ 0 \\ 5 &= 0\ \ 1\ \ 2 \\ 11 &= 1\ \ 0\ \ 2 \\ 16 &= 1\ \ 2\ \ 1. \end{aligned}$$

One of the two winning moves is to take $T_1 = 5$ chips from the pile of size 11, and $T_2 = 9$ chips from the pile of size 16.

For fixed $k \geq 2$, to move in $\textbf{Rim}_k$ [= R(estricted N)im_k] you first select integers $\ell \geq 0$ and T satisfying $1 \leq T < 2k^\ell$, and withdraw T chips from any pile. Then you have the option of withdrawing precisely k^ℓ chips from any pile, including the pile from which the T chips were taken. This option may be exercised up to a total of $k-2$ times in any move, possibly using the same pile more than once, provided $k \geq 4$. Observe that Rim_2 is Nim, and Rim_3 is Trim.

A position in Rim_k is a second-player win if and only if for each column in the corresponding **base k game array**, the sum modulo k of the entries is zero. As an illustration, the position in Rim_5 with piles of sizes 71, 135, 138, 176, and 252 (denoted by $[71, 135, 138, 176, 252]_5$) has the array

$$\begin{array}{rcl} 71 &=& 0\ \ 2\ \ 4\ \ 1 \\ 135 &=& 1\ \ 0\ \ 2\ \ 0 \\ 138 &=& 1\ \ 0\ \ 2\ \ 3 \\ 176 &=& 1\ \ 2\ \ 0\ \ 1 \\ 252 &=& 2\ \ 0\ \ 0\ \ 2 \\ \hline \text{Column sums modulo 5} &=& 0\ \ 4\ \ 3\ \ 2. \end{array}$$

One of the winning moves for the first player is to take 62 chips from the pile of size 71 and 50 chips from the pile of size 176. This is effected by setting $\ell = 2$, taking $T = 37$ chips from the pile of size 71, and then exercising the option three times, once with the pile of size 71 and twice with the pile of size 176. Can the reader find a winning move in which chips are withdrawn from four distinct piles?

We have proved that the Grundy number (in base k notation) of a position in Rim_k can be obtained from the base k game array simply by adding modulo k the entries in each column. (For the unfamiliar reader, an account of the Grundy number can be found in Chapter 11 of [3].) For example, $g([71, 135, 138, 176, 252]_5) = 0432$ (in base 5) $= 117$.

To see how this result can be useful, consider the disjunctive sum of the game $G = [13, 20, 114]_2$ and $H = [71, 135, 138, 176, 252]_5$. (In a disjunctive sum of two games, a player moves by selecting exactly one of the two games and then making a legal move in the selected game. The other game remains unchanged.) Since $g(G) = 107 \neq 117 = g(H)$, the first player to move can force a move in the disjunctive sum.

Observe that $g([13, 10, 114]_2) = g([13, 20, 108]_2) = 117$. Thus, a winning move in the disjunctive sum is to select G and to take either 10 chips from the pile of size 20 or 6 chips from the pile of size 114. Since $g([61, 135, 138, 176, 252]_5) = 107$, another winning move is to select H and to take 10 chips from the pile of size 71. There are still other winning moves. How many can you find?

To win misère Rim_k, move as you would in normal Rim_k until your move would leave all piles of size less than k; then, instead of leaving 0 mod k chips, leave 1 mod k chips.

In this article, Trim and Rim_k are our response to a problem tangential to the theory for Nim_k. Let us conclude by proposing a **new problem**. For each $k \geq 3$, find a reasonably natural take-away game which is a second-player win if and only if for each column in the corresponding base k game array, the sum modulo 2 of the entries is zero. More generally, do this when 2 is replaced by any integer $m \geq 2$ and $m \neq k$.

The author wishes to express his thanks to Professor T. S. Ferguson for several helpful insights.

REFERENCES

[1] C. L. Bouton (1902) Nim, a game with a complete mathematical theory, *Ann. Math.* **3**, 35-39.

[2] E. H. Moore (1910) A generalization of a game called nim, *Ann. Math.* **11**, 93-94.

[3] J. H. Conway (1976) *On Numbers and Games*, Academic Press, New York.