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BETTING SYSTEMS WHICH MINIMIZE THE PROBABILITY OF RUIN*

THOMAS S. FERGUSON†

1. Introduction. Games like roulette or craps are unfavorable in that the odds always favor the house. In such games, whatever your initial fortune, if you play long enough you are bound to be ruined, i.e., to go broke or bankrupt. Red dog, and—surprisingly—blackjack as treated by Thorp in his book *Beat the Dealer* [5], are examples of favorable games. Loosely speaking, games are favorable if there occasionally occur favorable situations. Most of the time, the odds are against the player and he bets lightly. But when the odds favor him, he bets heavily to more than make up for the fact that such situations occur rarely. As an example, suppose you are playing a sequence of games in which the probability that you win is either .4 or .6; in fact, you know that in 80 % of the games, chosen at random, your win probability is .4, while the remaining 20 % of the time your win probability is .6. You are allowed to make bets on these games on an even money basis, after you are told whether your win probability is .4 or .6. When the probability of win is .4, you will bet as small as you can, say the minimum bet of $1. Where the probability of win is .6, you will bet heavily, say $10. Table 1 shows that your average or expected winnings in 100 games using such a betting system is $24. This is in spite of the fact that you will win only about 44 % of the games. If you were to bet $1 on each game you would lose an average of $12 in every 100 games.

There is an advantage in betting $100 instead of $10 when the probability of win is .6. Such a bet will increase the *average* winnings in 100 games to $384. The drawback is that the player's resources are in fact limited, and making large bets in order to get large expected winnings may make the probability of eventual ruin correspondingly large. In §9 it will be seen that there is also a reason to bet $8.52 instead of $10 when the odds are favorable. Such a system of betting will not make you rich very fast; rather, it will make you rich more surely by minimizing in some sense the probability that you will eventually be ruined. (In the main models considered below, if a player is not ruined, his resources tend to infinity.) Betting heavily in favorable situations increases the expected winnings in favorable games, but it also increases the fluctuations of the resources. From the

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TABLE 1

P(win)	% of time	Bet	Winnings in 100 games
.4	80%	$1	−$16
.6	20%	$10	+$40
	Expected winnings		$24

point of view of survival—that is, of not going broke—it is desirable to keep these fluctuations small. The problem, then, is to strike a balance between large expected winnings and small fluctuations of the resources.

This paper is concerned with survival in favorable games. The basic models for favorable games are presented in §2. In §3 related work of other authors is reviewed. There the main objective is to maximize the expected utility of the fortune at the end of a fixed number of games. Exhibited is a large class of utility functions which lead to betting systems in which bets are proportional to the resources. In §4 a contrasting model is investigated which leads to betting systems which are independent of the resources. This model requires, however, the freedom to borrow unlimited amounts without interest. In §§5, 6, 7, a model is analyzed in which the only possible bets are $1 and $2, and where the problem is to minimize the probability of ruin. The special case when the win probabilities are chosen independently from a uniform distribution is solved completely, and the betting system which minimizes the probability of ruin is explicitly exhibited. The results of this case and several other considerations allow us to arrive at a reasonable conjecture as to the asymptotic (for large fortunes) form of the optimal betting system for two very general models. These results are presented in §§8 and 9. Finally in §10, the results of §§8, 9 are generalized by discounting the probabilities of future survival.

For a deep and rigorous approach to the problems associated with optimal gambling systems, the book by Dubins and Savage [3] is strongly recommended.

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2. The models. The *basic model* for the favorable games we consider is as follows. An individual is confronted with a sequence of completely independent games, $G_1, G_2, \cdots$, in which he may place bets on an even money basis. Before he places his bet in game G_j, he is told the probability p_j that he will win that game. Although he does not know the values of $p_{j+1}, p_{j+2}, \cdots$ when game G_j is played, he does know that $p_1, p_2, p_3, \cdots$

vary as if they were chosen independently from a distribution with known cumulative distribution function $F(p)$. His fortune (or capital, or resources) after game G_j is denoted by X_j; accordingly, X_0 denotes his initial fortune. In game G_j he may bet any amount between zero and his fortune X_{j-1} inclusive. His problem is to choose a betting system which is optimal with respect to some given criterion. In this paper we consider the criteria of large expected utility of X_n (§§3, 4) and of small probability of ruin §§5–10).

A *betting system* tells the individual exactly how much to bet in each game. Theoretically, the amount b_j designated to be bet in game G_j could depend on various extraneous variables such as his pulse or blood pressure after previous games; but we consider only those betting systems in which b_j is a function only of $X_0, p_1, X_1, p_2, \cdots, X_{j-1}, p_j$. The optimal betting systems $\{b_j\}$ derived later are optimal only within this class. Thus, a betting system is for us a sequence of measurable functions $\{b_j(x_0, p_1, x_1, p_2, \cdots, x_{j-1}, p_j)\}$ with the understanding that if the observed past fortunes have been $x_0, x_1, \cdots, x_{j-1}$ and the observed win probabilities have been $p_1, p_2, \cdots, p_j$, then the bet $b_j(x_0, p_1, \cdots, x_{j-1}, p_j)$ is made. The natural betting restriction is

$$0 \leqq b_j \leqq x_{j-1}. \tag{1}$$

The basic model is too simple to allow one to speak of the criterion of survival. Obviously, if the player bets nothing, his probability of ruin is zero. We consider two general models which are modifications of the basic model and which force the player to bet if he is to survive. In Model I there is a fixed cost to play each game, and in Model II there is a positive minimum bet. Model I, considered in §8, has application to behavioral science in which the cost may be interpreted as a cost of living. Model II, considered in §9, is a closer approximation to what actually happens in blackjack.

In this paper *a favorable game* is defined to be one in which there exists a betting system for which the probability of ruin is less than one for some value $x > 0$ of the initial fortune X_0. Let q_x denote the infimum over all choices of betting systems $\{b_j\}$ of the probability of ruin when the initial fortune is $X_0 = x$.

$$q_x = \inf_{\{b_j\}} P\{\text{ruin} \mid \{b_j\}, X_0 = x\}. \tag{2}$$

It is clear that q_x exists for all x. A quite general argument shows that in Model II,

$$q_{x+y} \leqq q_x q_y \tag{3}$$

for all x and y. This argument is based only on the assumption that a richer player can make all the bets that a poorer player can. If your fortune is $x + y$, you may put away y and play as if your fortune were x. The probability of losing these x can be made as close as desired to q_x. But if you lose these x, you still have y left. The probability of then losing these y can be made as close as desired to q_y. The probability of losing your fortune $x + y$ by this system is as close as desired to $q_x q_y$; but you can do at least this well using an arbitrary betting system so that (3) follows. In Model I, (3) is replaced by $q_{x+y} \leqq q_x q_{y-1}$.)

In particular, inequality (3) shows that the minimum probability of ruin in favorable games tends to zero at least exponentially fast; for if $q_x < 1$, then $q_{nx} \leqq q_x^n \to 0$ as $n \to \infty$. Inequality (3) also shows that q_x is a nonincreasing function of x.

In §6 a special problem in which the only possible bets are \$1 and \$2 is considered and it is shown that in this problem q_x does tend to zero exponentially fast. In both Model I and Model II, the actual betting system which minimizes the probability of ruin seems extremely difficult to compute even in special cases. Instead we conjecture that the probability of ruin tends to zero exponentially and then derive the asymptotic form of the optimal betting system. To be specific, we conjecture that in the models considered

$$q_x r^{-x} \to c \quad \text{as} \quad x \to \infty \tag{4}$$

for some $0 < r < 1$ and some $c > 0$.

A betting system for which the amount bet b_j in game G_j depends only on the present resources X_{j-1} and the present probability of win p_j is called a *Markov betting system* (since the resulting sequence $X_0, X_1, X_2, \cdots$ becomes a Markov process). For many choices of a criterion on which to judge betting systems, for example in maximizing the expected utility of X_n or in minimizing the probability of ruin, it is heuristically clear that in the search for good betting systems attention may be restricted to Markov betting systems. The author is not aware of any general theorems in the literature containing such an assertion. Proofs of such general theorems seem to encounter formidable measure theoretic difficulties. When the criterion is to minimize the probability of ruin, it would seem that one could go further and restrict attention to *stationary Markov betting systems* for which $b_j(x_{j-1}, p_j)$ is independent of j (so that $X_0, X_1, X_2, \cdots$ becomes a Markov process with stationary transition probabilities). Again, no general theorem along these lines seems to be available. In §5 a proof of this fact is given for the particular problem considered there. This proof is rather general but it does depend on the existence of an optimal first stage betting function $b_1(x_0, p_1)$, a fact easy to show for the problem in §5. Indeed, this proof should work for Models I and II as well, but showing the existence of an optimal first stage betting function is much more difficult. This difficulty may be overcome by showing that q_x is a continuous function of x. A. J. Truelove [6] has proved the continuity of q_x in Model I in the special case where F is degenerate. It is possible that his proof may extend to the general case of Model I and to Model II. In §8 we assume in addition to the conjecture (4) that q_x is continuous, and outline the derivation of the asymptotic form of optimal betting system.

3. Proportional betting systems. In this section we consider the basic model and try to maximize the utility of our fortune n steps ahead. No matter what the utility function is, the optimal betting system, if it exists, will turn out to be Markov. For Markov betting systems, we use the notation $b_j(x, p)$ to represent the amount bet in game G_j if $X_{j-1} = x$ and $p_j = p$.

If we are interested only in having large expected resources at the end of n games, that is, if we want to maximize $E(X_n \mid X_0)$, then we would employ the Markov betting system

$$b_j(x, p) = \begin{cases} x & \text{if } p > \frac{1}{2}, \\ 0 & \text{if } p \leqq \frac{1}{2}, \end{cases} \tag{5}$$

for $j = 1, 2, \cdots, m$. From other points of view, however, this betting system is not very good. If $F(\frac{1}{2}) < 1$ (we take F to be continuous from the right), then the user of such a betting system is almost certain to be eventually ruined. The trouble arises from the fact that the utility of money to a player is not necessarily linear in the amount. It is of interest to consider a larger class of utility functions of the form

$$U_\alpha(x) = \frac{x^\alpha - 1}{\alpha}, \tag{6}$$

since it turns out that the betting systems which maximize $E(U_\alpha(X_n) \mid X_0)$ are also quite simple. Kelly [4], Bellman and Kalaba (see [1, Chap. 17 and the references cited on p. 230]) and Breiman [2] have considered the important special case, corresponding to $\alpha = 0$, of maximizing $E(\log X_n \mid X_0)$. The Markov betting system

$$b_j(x, p) = \begin{cases} (2p - 1)x & \text{if } p > \frac{1}{2}, \\ 0 & \text{if } p \leqq \frac{1}{2}, \end{cases} \tag{7}$$

which we shall refer to as Kelly's betting system, maximizes $E(\log X_n \mid X_0)$. This betting system and, more generally, those which maximize $E(U_\alpha(X_n) \mid X_0)$ are remarkably simple in that they are *proportional stationary Markov betting systems* (i.e., $b_j(x, p) = \pi(p)x$, with $0 \leqq \pi(p) \leqq 1$).

That a betting system is optimal with respect to some arbitrarily chosen utility function is not a very convincing argument in favor of its use. It is preferable to consider criteria which are more intrinsic. Breiman [2] in a problem closely related to this one considers two such criteria, one being minimal expected time to reach a fixed level of resources, and the other being maximal rate at which the resources get large. For both criteria, Kelly's betting system turns out to be asymptotically optimal. That a particular betting system is to be preferred on the basis of several different criteria is a strong argument for its use in more general situations. Kelly's betting system may be endorsed from this point of view.

4. An analogous model. Let us consider a modified model—the Billie Sol Estes model—in which the gambler is allowed to bet amounts larger than his resources, and in which, in fact, his resources are allowed to be negative. As before, he is to play a sequence of games $G_1, G_2, \cdots$ with respective probabilities $p_1, p_2, \cdots$ of his winning and with initial resources X_0. Instead of the betting restriction (1), this time we require only

$$0 \leqq b_j, \tag{8}$$

namely, that he bet nonnegative amounts. Instead of taking utility functions of the form (6), we take them of the form

$$V_\theta(x) = -e^{-\theta x}, \tag{9}$$

where $\theta > 0$. These utility functions lead to simple betting systems which differ in an important respect from the proportional betting systems. The Markov betting system

$$b_j(x, p) = \begin{cases} \frac{1}{2\theta} \log \frac{p}{1-p} & \text{if } p > \frac{1}{2}, \\ 0 & \text{if } p \leqq \frac{1}{2}, \end{cases} \tag{10}$$

for $j = 1, 2, \cdots, n$, maximizes $E(V_\theta(X_n) \mid X_0)$. This is a stationary Markov betting system and, far from being a proportional betting system, it is independent of x. You don't have to keep track of how much your resources are to be able to tell how much you are going to bet!

The proof that the betting system (10) maximizes $E(V_\theta(X_n) \mid X_0)$ is essentially the same as that given by Bellman and Kalaba (see [1, pp. 223–224]) of the fact that Kelly's betting system maximizes $E(U_0(X_n) \mid X_0)$. The idea of this proof is as follows. It is easily checked that the $b_n(x, p)$ of (10) maximizes $E(V_\theta(X_n) \mid X_{n-1})$, and that, luckily, the maximum value turns out to be

$$\max_{b_n} E(V_\theta(X_n) \mid X_{n-1}) = -re^{-\theta X_{n-1}},$$

where

$$r = \int_0^{\frac{1}{2}} dF(p) + \int_{\frac{1}{2}}^1 2\sqrt{p(1-p)}\, dF(p). \tag{11}$$

On the previous step, maximizing $E(V_\theta(X_n) \mid X_{n-2})$ is equivalent to maximizing $rE(V_\theta(X_{n-1}) \mid X_{n-2})$, which not only gives for $b_{n-1}(x, p)$ the same solution (10), but also gives the maximum value of $E(V_\theta(X_n) \mid X_{n-2})$ as $r^2V_\theta(X_{n-2})$. This obviously may be continued to prove (10) for j down to 1. (Actually, as Bellman and Kalaba point out, it is not necessary that the p_j be chosen independently or even randomly. In fact, even if the gambler is told the exact values of the p_j before he starts to play, the betting system (10) is still optimal!)

(Another side remark: A closer analogue to the utility functions U_α of (3) would be the utility functions, defined for all real values of θ, $V_\theta^*(x) = (1 - e^{-\theta x})/\theta$ for $\theta \neq 0$, and $V_0^*(x) = x$. For $\theta > 0$, these utility functions are equivalent to those of (9). An optimal betting system generalizing (10) may be worked through provided the betting restriction (8) is replaced by $c_1 \leqq b_j \leqq c_2$, with c_1 and c_2 independent of x.)

The main reason we discussed this model is that analogous to the results of Breiman for Kelly's betting system in the basic model, we claim there is a more intrinsic criterion than the arbitrary utility function (9) for which the betting system (10) is asymptotically optimal for some value of θ. That criterion is survival. In Models I and II this claim is based on the conjecture (4) that the probability of ruin tends to zero exponentially as the fortune tends to infinity, so that the utility function (9) will be approximately valid for some value of θ when the fortune is large.

5. A discrete model. In this section, we make the assumptions of the basic model with the additional restriction that the only allowable bets are \$1 and \$2. With this restriction, we may as well assume also that the resources x are an integer number of dollars.

The betting system giving the largest expected gain per play is the stationary Markov betting system $b_j = 1$ if $p_j \leqq \frac{1}{2}$ and $b_j = 2$ if $p_j > \frac{1}{2}$, for $X_{j-1} = x \geqq 2$. (If $X_{j-1} = 1$, then $b_j \equiv 1$, since bets larger than the resources are forbidden.) This maximum expected gain for resources $x \geqq 2$ is

$$\int_0^{\frac{1}{2}} (2p-1)\, dF(p) + 2\int_{\frac{1}{2}}^1 (2p-1)\, dF(p)$$
$$= 2(2\mu - 1) + \int_0^{\frac{1}{2}} (1-2p)\, dF(p),$$

where μ is the first moment of the distribution F. If this expected gain is not positive, then $X_0, X_1, X_2, \cdots$ forms a lower semimartingale bounded by zero, so that with probability one, X_n converges to some random variable X_∞. But X_n does not converge unless it is zero from some point on. Hence, the probability of eventual ruin is one. On the other hand, if this expected gain is positive, then this stationary Markov betting system itself gives its user a positive probability of survival, whatever his initial resources (i.e., $q_1 < 1$). Hence we have the following.

LEMMA 1. *In order that the game be favorable it is necessary and sufficient that*

$$2(2\mu - 1) + \int_0^{\frac{1}{2}} (1 - 2p)\, dF(p) > 0. \tag{12}$$

This condition is obviously satisfied if, for example, $\mu > \frac{1}{2}$.

We present a brief lemma which states that no matter what betting system is used, the probability is one that either X_n is zero from some point on or $X_n \to \infty$. This implies that minimizing the probability of ruin is the same as maximizing the probability that the fortune tends to infinity.

LEMMA 2. *For every finite* t, $P\{0 < X_n < t \text{ i.o.}\} = 0$ (i.o. = *infinitely often*).

Proof. The proof we give depends only on the fact that there is an $\epsilon_t > 0$ such that for all betting systems and for all n and $j \leqq t$, $P\{X_{n+t} = 0 | X_n = j\} \geqq \epsilon_t$. In our specific problem, we may take $\epsilon_t = P\{\text{loss } t \text{ consecutive times betting \$1 each time}\} = \theta^t$, where $\theta = \int (1 - p)\, dF(p)$. If F is degenerate at 1, the lemma is obvious; otherwise, $\theta > 0$.

Let N_1 be the first n such that $0 < X_n < t$; if no such n exists, let $N_1 = +\infty$. By induction, define N_{i+1} as the first n larger than $N_i + t$ such that $0 < X_n < t$ provided $N_i < \infty$ and such an n exists; otherwise define N_{i+1} as $+\infty$. The sets $\{N_i < \infty\}$ are decreasing in i to the limit $\bigcap\{N_i < \infty\} = \{0 < X_n < t \text{ i.o.}\}$ as $i \to \infty$. But,

$$\begin{aligned} P\{N_{i+1} < \infty\} &= P\{N_{i+1} < \infty \mid N_i < \infty\}\, P\{N_i < \infty\} \\ &= \sum_{j=1}^{t-1} P\{N_{i+1} < \infty \mid X_{N_i} = j\} P\{X_{N_i} = j \mid N_i < \infty\} P\{N_i < \infty\} \\ &\leqq (1 - \epsilon_t) P\{N_i < \infty\}. \end{aligned}$$

Hence, $P\{N_{i+1} < \infty\} \leqq (1 - \epsilon_t)^i P\{N_1 < \infty\} \to 0$ as $i \to \infty$, completing the proof.

We shall suppose from now on that the condition of Lemma 1 is satisfied. We first derive the equation which an optimal betting system and associated probabilities of ruin must satisfy. This equation is the fundamental functional equation whose use is championed by R. Bellman.

LEMMA 3. *For the* q_x *defined by* (2),

$$q_x = \int \{ \inf_b [pq_{x+b} + (1 - p)q_{x-b}]\} \, dF(p), \tag{13}$$

where the infimum is taken over the functions $b(x, p)$ *which take only those values* 1 *or* 2.

Proof. That q_x is less than or equal to the integral on the right hand side is clear since this integral is the infimum of the probability of ruin given $X_0 = x$ over all $\{b_j\}$ for which $b_2, b_3, \cdots$ do not depend on x_0 and p_1.

To show the reverse inequality, let $\{b_j^\epsilon\}$ be ϵ-optimal in the sense that

$$P\{\text{ruin} \mid \{b_j^\epsilon\}, X_0 = x\} \leqq q_x(1 + \epsilon).$$

Note that for all x_0, p, and x_1,

$$P\{\text{ruin} \mid \{b_j^\epsilon\}, X_0 = x, p_1 = p, X_1 = x_1\} \geqq q_{x_1}.$$

Then

$$\begin{aligned}\int \{\inf_b [pq_{x+b} + (1 - p)\, q_{x-b}]\} \, dF(p) &\leqq \int [pq_{x+b_1^\epsilon} + (1 - p)\, q_{x-b_1^\epsilon}] \, dF(p) \\ &\leqq P\{\text{ruin} \mid \{b_j^\epsilon\}, X_0 = x\} \leqq q_x(1 + \epsilon)\end{aligned}$$

for all x, completing the proof.

It is clear for the problem under consideration that there exists a betting function $b^*(x, p)$ which minimizes $pq_{x+b^*} + (1 - p)q_{x-b^*}$, where b^* takes only the values 1 or 2. Let $\{b_j^*\}$ be that stationary Markov betting system such that $b_j^* = b^*$ for all j, and let $q_x^* = P\{\text{ruin} \mid \{b_j^*\}, X_0 = x\}$. It is our objective to show that $q_x = q_x^*$, thus showing that $\{b_j^*\}$ is an optimal betting system.

Let $\epsilon > 0$ and choose a betting system $\{b_j^\epsilon\}$ such that

$$q_x^\epsilon = P\{\text{ruin} \mid \{b_j^\epsilon\}, X_0 = x\} \leqq q_x(1 + \epsilon)$$

for all x. Let $\{b_j^{(n)}\}$ be that betting system which uses b^* n times followed by $\{b_j^\epsilon\}$, i.e., for $j \leqq n$, $b_j^{(n)} = b_j^*$, and for $j > n$, $b_j^{(n)} = b_{j-n}^\epsilon$ in the sense that

$$b_j^{(n)}(x_0, p_1, \cdots, x_{j-1}, p_j) = b_{j-n}^\epsilon(x_n, p_{n+1}, \cdots, x_{j-1}, p_j).$$

Let $q_x^{(n)} = P\{\text{ruin} \mid \{b_j^{(n)}\}, X_0 = x\}$.

LEMMA 4. $$q_x^{(n)} \leqq q_x(1 + \epsilon).$$

Proof. The proof is by induction. Since $\{b_j^{(0)}\} = \{b_j^\epsilon\}$, the lemma is valid for $n = 0$. Suppose it is true for $n = k$. Then since $\{b_j^{(k+1)}\}$ is b^* followed by $\{b_j^{(k)}\}$,

$$\begin{aligned} q_x^{(k+1)} &= \int [pq_{x+b^*}^{(k)} + (1-p)\, q_{x-b^*}^{(k)}]\, dF(p) \\ &\leqq \int [pq_{x+b^*} + (1-p)\, q_{x-b^*}]\, dF(p)\, (1+\epsilon) \\ &= q_x(1+\epsilon), \end{aligned}$$

completing the induction.

THEOREM 1.

$$q_x = q_x^*.$$

Proof. That $q_x \leqq q_x^*$ is obvious from the definition of q_x. To show the reverse inequality, note that $P\{X_n = 0 \mid \{b_j^*\}, X_0 = x\}$ is nondecreasing in n and converges to q_x^* as $n \to \infty$. Consequently, for n sufficiently large,

$$\begin{aligned} q_x^*(1-\epsilon) &\leqq P\{X_n = 0 \mid \{b_j^*\}, X_0 = x\} \\ &= P\{X_n = 0 \mid \{b_j^{(n)}\}, X_0 = x\} \\ &\leqq q_x^{(n)} \leqq q_x(1+\epsilon). \end{aligned}$$

The validity of this inequality for all $\epsilon > 0$ implies $q_x^* \leqq q_x$, completing the proof.

To find an optimal betting system in terms of q_x is now an easy matter. Since b^* is that number, one or two, which minimizes $p\, q_{x+b^*} + (1-p)q_{x-b^*}$, it is optimal to bet 1 if $p\, q_{x+1} + (1-p)q_{x-1} \leqq p\, q_{x+2} + (1-p)q_{x-2}$, and to bet 2 otherwise. Thus, when the resources are x, and the probability of win is p, it is optimal to bet 1 if $p \leqq c_x$ and 2 if $p > c_x$, where $c_1 = 1$ (one cannot bet more than one has) and

$$c_x = \frac{(q_{x-2} - q_{x-1})}{(q_{x-2} - q_{x-1}) + (q_{x+1} - q_{x+2})}, \qquad x = 2, 3, \cdots \tag{14}$$

(note that $q_0 = 1$). Since the game is assumed favorable, (3) implies that $0 < c_x < 1$, for $x \geqq 2$.

To complete the solution, there still remains the problem of describing a method whereby the ruin probabilities q_x may be computed. Equation (13) provides a set of equations to be satisfied by the q_x. To write these equations in more convenient form we introduce the notation

$$\mu(y) = \int_0^y p\, dF(p).$$

so that $\mu(1) = \mu$, the mean of the distribution F. Recall that F is taken to be continuous from the right and so $\mu(y)$ is also continuous from the right (since $\int_0^y p\, dF(p) = \int_0^{y+} p\, dF(p)$). Simple manipulations reduce (13) to

$$(15)\quad \begin{aligned} q_x = \mu\, q_{x+2} + (1 - \mu)q_{x-2} - F(c_x)(q_{x-2} - q_{x-1}) \\ + \mu(c_x)(q_{x-2} - q_{x-1} + q_{x+1} - q_{x+2}) \end{aligned}$$

for $x = 2, 3, \cdots$, where c_x satisfies (14). This set of equations is to be solved for the q_x subject to the boundary conditions: (a) $q_x > q_{x+1}$ for all x, (b) $q_1 - \mu q_2 = 1 - \mu$, (c) $q_0 = 1$, and (d) $\lim_{x\to\infty} q_x = 0$. Equation (15) is a fourth order recurrence relation; given q_{x-2}, q_{x-1}, q_x, and q_{x+1}, one can solve for q_{x+2}. By two successive changes of variable, this can be reduced to a second order recurrence relation. First, let

$$(16)\qquad \alpha_x = q_{x-1} - q_x, \qquad x = 1, 2, \cdots.$$

Equations (15) become

$$(17)\quad \begin{aligned} \mu(c_x)(\alpha_{x-1} + \alpha_{x+2}) - F(c_x)\alpha_{x-1} \\ - (\alpha_{x-1} + \alpha_x + \alpha_{x+1} + \alpha_{x+2})\mu + (\alpha_{x-1} + \alpha_x) = 0 \end{aligned}$$

for $x = 2, 3, \cdots$, where $c_x = \alpha_{x-1}/(\alpha_{x-1} + \alpha_{x+2})$, subject to the boundary conditions: (a) $\alpha_x > 0$ for all x, (b) $(1 - \mu)\alpha_1 = \mu\alpha_2$, and (c) $\sum_1^\infty \alpha_x = 1$. Second, let

$$(18)\qquad \beta_x = \frac{\alpha_{x+1}}{\alpha_x}, \qquad x = 1, 2, \cdots.$$

Equations (17) become

$$(19)\quad \begin{aligned} \mu(c_x)(1 + \beta_{x-1}\beta_x\beta_{x+1}) + (1 + \beta_{x-1}) \\ = F(c_x) + (1 + \beta_{x-1} + \beta_{x-1}\beta_x + \beta_{x-1}\beta_x\beta_{x+1})\mu \end{aligned}$$

for $x = 2, 3 \cdots$, where $c_x = (1 + \beta_{x-1}\beta_x\beta_{x+1})^{-1}$ subject to the boundary conditions: (a) $\beta_x > 0$ and (b) $\beta_1\mu = 1 - \mu$.

To solve (19) subject to the stated boundary conditions is indeed a difficult problem. One possible approach is illustrated in the next section where F is the uniform distribution over the interval [0, 1]. It seems likely, however, that it is more feasible in general to approximate the q_x directly by the minimum probability of being ruined in the first n steps when the initial resources are x. This corresponds to a dynamic programming problem, refinements of which have been carried out by A. J. Truelove [6] in a special case of the more complex Model I.

6. A special case, completely solved. Here, we treat the model of the previous section under the assumption that F is the uniform distribution on the interval [0, 1]. It is easily checked that condition (12) is satisfied so that the game is indeed favorable. Equations (19) reduce to

$$\beta_{x-1}\beta_x^2\beta_{x+1}^2 = (1-\beta_x)(1+\beta_{x-1}\beta_x\beta_{x-1})$$

for $x = 2, 3, \cdots$, subject to the boundary conditions (a) $\beta_x > 0$ and (b) $\beta_1 = 1$. This quadratic equation in β_{x+1} has as its only positive solution

$$\beta_{x+1} = \frac{1-\beta_x}{\beta_x}\left(1+\left(1+\frac{4}{\beta_{x-1}(1-\beta_x)}\right)^{\frac{1}{2}}\right) \tag{20}$$

for $x = 2, 3, \cdots$. Given $\beta_1 = 1$ and an arbitrary β_2, we may compute $\beta_3, \beta_4, \cdots$ successively from (20). The problem is to choose a value of β_2 for which the boundary condition (a) is satisfied. We know from the problem which led to (20) that there exists at least one such value of β_2. It will be shown below that there exists at most one such value. For this purpose, the following lemmas prove useful. Let

$$f(x, y) = \frac{1-y}{2y}\left(1+\sqrt{1+\frac{4}{x(1-y)}}\right), \tag{21}$$

so that $\beta_{k+1} = f(\beta_{k-1}, \beta_k)$. We will need later the two partial derivatives,

$$\frac{\partial}{\partial x}f(x, y) = -x^{-2}y^{-1}\left(1+\frac{4}{x(1-y)}\right)^{-\frac{1}{2}},$$

$$\frac{\partial}{\partial y}f(x, y) = -\frac{1}{2y^2}\left[1+\left(1+\frac{4}{x(1-y)}\right)^{\frac{1}{2}}\left(\frac{(x+2)(1-y)+2}{x(1-y)+4}\right)\right],$$

which show that $f(x, y)$ is a decreasing function of both x and y in the interval $0 < x < 1, 0 < y < 1$.

LEMMA 5. (i) $\beta_x > 1/\sqrt{2}$ *for all* $x \geqq 1$.
(ii) $\beta_x < 2(\sqrt{2}-1)$ *for all* $x \geqq 2$.

Proof. (i) First note that (19) automatically implies that all $\beta_x < 1$ for $x \geqq 2$. Suppose contrary to (i) that $\beta_x \leqq 1/\sqrt{2}$ for some x. Then

$$\beta_{x+1} = f(\beta_{x-1}, \beta_x) \geqq f\left(1, \frac{1}{\sqrt{2}}\right) = 1;$$

contradiction.

(ii) Suppose to the contrary that $\beta_x > 2(\sqrt{2}-1)$ for some $x \geqq 2$. Then

$$\beta_{x+1} = f(\beta_{x-1}, \beta_x) \leqq f\left(\frac{1}{\sqrt{2}}, 2(\sqrt{2}-1)\right) = \frac{1}{\sqrt{2}},$$

contradicting (i), and completing the proof.

LEMMA 6. *For* $1/\sqrt{2} \leqq x \leqq 1$ *and* $1/\sqrt{2} \leqq y \leqq 2(\sqrt{2}-1)$,

$$\text{(i)} \quad \frac{\partial}{\partial x} f(x, y) > -\frac{3}{4} \quad \textit{and} \quad \text{(ii)} \quad \frac{\partial}{\partial y} f(x, y) < -2.$$

Proof.

$$\text{(i)} \quad \frac{\partial}{\partial x} f(x, y) \geqq -x^{-2} y^{-1} \left(1 + \frac{4}{1-y}\right)^{-\frac{1}{2}}$$

$$\geqq -2\sqrt{2}\left(1 + \frac{4\sqrt{2}}{1-\sqrt{2}}\right)^{-\frac{1}{2}} = \frac{-(8-2\sqrt{2})}{7} > -\frac{3}{4}$$

$$\text{(ii)} \quad \frac{\partial}{\partial y} f(x, y) \leqq -\frac{1}{8(\sqrt{2}-1)^2}\left[1 + \left(1 + \frac{4}{x(1-y)}\right)^{\frac{1}{2}}\right.$$

$$\left.\cdot\left(\frac{x(1-y)+2}{x(1-y)+4}\right)\right] \leqq (\text{same evaluated at } x = 1 \text{ and } y = 1/\sqrt{2})$$

$$= -\frac{31 + 22\sqrt{2}}{28} < -2.$$

The following theorem shows that the recurrence relation (20) is not stable; that is, a small error in β_2 will lead to a larger error in β_3, and an even larger error in β_4, etc.

THEOREM 2. *Put $\beta_1 = 1$ and let β_3, β_4, $\cdots$ be functions of β_2 through the recurrence relation* (19). *Then, if $1/\sqrt{2} < \beta_k < 2(\sqrt{2} - 1)$ for $2 \leqq k \leqq x$,*

$$\frac{d\beta_{x+1}}{d\beta_x} < -\frac{3}{2}, \qquad x = 2, 3, \cdots.$$

Proof. By induction: it is obviously valid for $x = 2$ since $d\beta_3/d\beta_2 = \partial f(\beta_1, \beta_2)/\partial\beta_2 < -2 < -3/2$. Now suppose it is valid for $x < N$; then,

$$\frac{d\beta_{N+1}}{d\beta_N} = \frac{\partial}{\partial\beta_{N-1}} f(\beta_{N-1}, \beta_N) \frac{d\beta_{N-1}}{d\beta_N} + \frac{\partial}{\partial\beta_N} f(\beta_{N-1}, \beta_N) < \frac{3}{4}\cdot\frac{2}{3} - 2 = -\frac{3}{2},$$

completing the proof.

This theorem serves two purposes. First, it shows that there is at most one solution to (20) subject to conditions (a) and (b). For, if $\beta_1^* = 1$, β_2^*, β_3^*, $\cdots$ is a solution, then a choice of $\beta_2 = \beta_2^* + \epsilon$, $\epsilon > 0$, will lead to $\beta_3 < \beta_3^* - \frac{3}{2}\epsilon$, $\beta_4 > \beta_4^* + (\frac{3}{2})^2\epsilon$, $\beta_5 < \beta_5^* - (\frac{3}{2})^3\epsilon$, etc. (similarly, $\beta_2 = \beta_2^* - \epsilon$, $\epsilon > 0$, will lead to $\beta_3 > \beta_3^* + \frac{3}{2}\epsilon$, $\beta_4 < \beta_4^* - (\frac{3}{2})^2\epsilon$, etc.) until eventually the assertion of Lemma 5 that $1/\sqrt{2} < \beta_x < 2(\sqrt{2} - 1)$ is violated.

Second, it is the basis of a simple algorithm for approximating the solution to (20) subject to conditions (a) and (b) as closely as desired. The theorem implies that the computed values of the β_x will be alternatively above and below their true values. A trial choice of β_2 will lead to β_3, β_4, $\cdots$ until $1/\sqrt{2} < \beta_x < 2(\sqrt{2} - 1)$ is first violated at $x =$ some N. If N is even and

TABLE 2

β_1 =	1.000000
β_2 =	.768382
β_3 =	.794936
β_4 =	.791522
β_5 =	.791955
β_6 =	.791900
β_7 =	.791908
β_8 =	.791906

$\beta_N \leqq 1/\sqrt{2}$, or if N is odd and $\beta_N \geqq 2(\sqrt{2} - 1)$, then β_2 was chosen too small. Similarly, if N is even and $\beta_N \geqq 2(\sqrt{2} - 1)$, or if N is odd and $\beta_N \leqq 1/\sqrt{2}$, then β_2 was chosen too large. Hence, by continually bisecting values of β_2 known to be too small and too large, a sequence of values of β_2 may be obtained which converges exponentially to the true value of β_2. (Simple improvements of this method will converge much faster—about as fast as Newton's method converges to the root of an equation—but a description of these seem out of place here.)

On the basis of some such method, the first few values of the β_x were computed to six decimals accuracy, and are given in Table 2.

Using the β_x, we must compute the values of the α_x from (18) subject to condition (c): $\sum_1^\infty \alpha_x = 1$. It is clear that $\alpha_{x+1} = \alpha_1 \prod_1^x \beta_k$, where α_1 is chosen so that condition (c) is satisfied. But $\sum_1^\infty \alpha_x$ involves knowledge of the infinite sequence of the β_x, not just the first few terms. We will show that the rest of the β_x may well be approximated to six decimals by the value of β_8. We will show, in fact, that the β_x converge to a constant.

If the β_x do converge to a constant r, then it is easy to find the value of r. The function $f(x, y)$ of (21) is clearly continuous for (x, y) within the unit square, so that $r = f(r, r)$. Roots of the equation are also roots of the fifth degree equation $r^5 + r^4 - r^3 + r - 1 = 0$. There is one and only one root of this equation in the unit interval [0, 1], this being $r = .7919064293$ to ten decimals. We now demonstrate that the β_x do indeed converge to r, the root of the equation $r = f(r, r)$. We first establish the fact, indicated by Table 2, that the true values of the β_x lie alternatively above and below r.

LEMMA 7. *If* $\beta_x < r$, *then* $\beta_{x+1} > r$. *If* $\beta_x > r$, *then* $\beta_{x+1} < r$.

Proof. Suppose $\beta_x < r$. We will show that a choice of $\beta_{x+1} = r$ is too small. If $\beta_{x+1} = r$, then all succeeding $\beta_{x+2}, \beta_{x+3}, \cdots$ must equal r also since $f(r, r) = r$ and there is at most one solution. But $\beta_{x+2} = f(\beta_x, \beta_{x+1}) > f(r, r) = r$. Hence, β_{x+2n} are too large, and β_{x+2n-1} are too small. Hence, the true value of β_{x+1} must be greater than r. Similarly, if $\beta_x > r$.

THEOREM 3.

$$|r - \beta_{x+1}| \leqq \Delta |r - \beta_x|$$

where $\Delta = \sqrt{3/4} < 1$.

TABLE 3

x	α_x	q_x	c_x
1	.17536	.82464	1.00000
2	.17536	.64929	.62080
3	.13474	.51454	.67409
4	.10711	.40743	.66742
5	.08478	.32266	.66827
6	.06714	.25551	.66816
7	.05317	.20234	.66818
8	.04211	.16023	.66817

Proof. Suppose $\beta_x < r$ and $\beta_{x+1} > r + \Delta(r - \beta_x)$. Then $\beta_x > r - \Delta^{-1}(\beta_{x+1} - r)$, so that $\beta_{x+2} = f(\beta_x, \beta_{x+1}) < f(r - \Delta^{-1}(\beta_{x+1} - r), r + (\beta_{x+1} - r))$. But since

$$\frac{d}{dx} f(r - \Delta^{-1} x, r + x) = -\Delta^{-1} f_x(r - \Delta^{-1} x, r + x) + f_y(r - \Delta^{-1} x, r + x) < \Delta^{-1} \frac{3}{4} - 2 = -(2 - \Delta),$$

and $f(r, r) = r$, it must be that $\beta_{x+2} < r - (2 - \Delta)(\beta_{x+1} - r)$. Hence, β_{x+2} must be further away from r than β_{x+1} by a factor of $(2 - \Delta) > 1$. Inductively, β_{x+k+1} is also further away from r than β_{x+k} by a factor of $2 - \Delta$. It is clear then that β_{x+k} is too small for even k and too large for odd k. Thus β_{x+1} was chosen too large; i.e., $\beta_{x+1} - r > \Delta(r - \beta_x)$.

A similar proof works if $\beta_x > r$.

This theorem shows not only that the β_x converge to r, but that the convergence is monotone in absolute value and at an exponential rate. Thus we may use β_8 as an approximation good to six decimals of β_9, β_{10}, etc. Using Table 2, the values of α_x, q_x, and c_x are now readily computed.

From Table 3 it may be guessed that c_x tends to a constant. This is easily seen since $c_x = (1 + \beta_{x-1}\beta_x\beta_{x+1})^{-1} \to (1 + r^3)^{-1} = .6681736 \cdots$. Thus, for x large the optimal strategy is to bet \$2 if $p > .6681736 \cdots$ and to bet \$1 if $p < .6681736 \cdots$. Furthermore, the actual optimal strategy is seen in Table 3 to converge very rapidly to this one.

It is interesting to contrast this betting system with the one which for $x \geqq \$2$ bets \$2 if $p > \frac{1}{2}$ and \$1 if $p \leqq \frac{1}{2}$. The probabilities of ruin tend to zero exponentially at rate .812 (rather than .792) and the first few values of q_x are $q_1 = .837$, $q_2 = .675$, $q_3 = .549$, and $q_4 = .446$.

7. Asymptotic solution for general F. Presumably, some method like that of the previous section for approximating the solution will work in the general case where F satisfies the condition of Lemma 1. It is easy, however, to get an approximation to the optimal betting system for large resources in the general case, provided the conjecture (4) on the exponential rate of convergence of the probability of ruin to zero is valid. In the present case, we employ the weaker conjecture: that for some $0 < r < 1$,

$$\beta_x \to r \quad \text{as} \quad x \to \infty. \tag{22}$$

Under this condition,

$$c_x \to (1 + r^3)^{-1} \quad \text{as} \quad x \to \infty, \tag{23}$$

and since $F(c_x) - c_x^{-1}\mu(c_x)$ is a continuous function of c_x (though neither of the terms separately is continuous), (19) converges to

$$\begin{aligned} F((1 + r^3)^{-1}) - (1 + r^3)\mu((1 + r^3)^{-1}) \\ + (1 + r + r^2 + r^3)\mu - (1 + r) = 0. \end{aligned} \tag{24}$$

We would like to show there is a unique root r of this equation, $0 < r < 1$, since then, by finding this root, we would obtain not only the exponential rate of convergence of the probability of ruin to zero, but also the approximate optimal betting system for large resources by (23).

Denoting the left side of (24) by $g(r)$, we have $g(0) = 0$ and $g(1) = F(\frac{1}{2}) - 2\mu(\frac{1}{2}) + 4\mu - 2 > 0$, since we are assuming that F satisfies (12). The derivative of g with respect to r exists at all points such that F does not give positive mass to $(1 + r^3)^{-1}$, and

$$g'(r) = 3r^2 \int_{(1+r^3)^{-1}}^{1} p\, dF(p) + (1 - 2r)\, \mu - 1,$$

so that $g'(0) = \mu - 1 < 0$. It is apparent that $g'(r)$ is an increasing function of r, and that $g(r)$ is a continuous function of r. This implies that $g(r)$ is convex, and hence has a unique root between zero and one.

In summary, if the conjecture (22) is valid, then r may be found from (24), and for large resources, the optimal betting system is (approximately) to bet \$1 if $p < (1 + r^3)^{-1}$ and to bet \$2 if $p > (1 + r^3)^{-1}$.

8. Model I. The general problem for Model I is investigated in this section by a method analogous to that used in the preceding section. It is useful to write $q(x)$ instead of q_x to represent the minimal probability of ruin when the resources are x (equation (2)).

Recall that in Model I, the gambler pays one unit of resources to play each game. For the sake of convenience, X_{j-1} is taken to represent his resources after having paid the fee to play game G_j for $j = 1, 2, \cdots$. In this way the betting restriction is still

$$0 \leqq b_j \leqq x_{j-1}. \tag{1}$$

There is a very simple necessary and sufficient condition that a game of

Model I be favorable, namely, that F give positive mass to the half-open interval $(\frac{1}{2}, 1]$. For if so, there exists a number M such that the betting function

$$b(x, p) = \begin{cases} 0 & \text{if } p \leqq \frac{1}{2}, \\ M & \text{if } p > \frac{1}{2}, \end{cases}$$

has expected gain greater than one. After the cost of living of one unit is deducted, the net expected gain is still positive. From any initial fortune $X_0 > 1$, there is a positive probability that the gambler will eventually have fortune greater than M; and then there is a further positive probability that this fortune will never get below M, provided he uses the above betting function thereafter. The condition that $F(\frac{1}{2}) < 1$ is also necessary and sufficient that a game of Model II be favorable.

The Bellman functional equation corresponding to (13) for Model I problems is

$$q(x) = \int_0^1 \inf_{0 \leqq b \leqq x} \varphi(x, p, b)\, dF(p), \tag{25}$$

where

$$\varphi(x, p, b) = p\, q(x + b - 1) + (1 - p)\, q(x - b - 1) \tag{26}$$

represents the infimum probability of ruin given $X_0 = x$, $p_1 = p$, and a bet b is made in game G_1. (For negative x, $q(x)$ is defined to be one.) If it is true that $q(x)$ is a continuous function of x (a hypothesis discussed in §2), then there clearly exists a function $b^*(x, p)$ such that $\varphi(x, p, b^*) \equiv \inf_{0 \leqq b \leqq x} \varphi(x, p, b)$. With the existence of such a function, the analysis carried out in §5 can be modified to show that the stationary Markov betting system which uses $b^*(x, p)$ at each stage of the betting is optimal.

We shall now demonstrate how the conjecture that, for some $0 < r < 1$ and $c > 0$,

$$q(x)r^{-x} \longrightarrow c \quad \text{as} \quad x \longrightarrow \infty \tag{4}$$

may be used to find the value of r, and the limit of the optimal betting function $b^*(x, p)$ as $x \longrightarrow \infty$.

Let $\epsilon > 0$ and find a number x_ϵ, whose existence is entailed by the conjecture, such that $x > x_\epsilon$ implies $| q(x)r^{-x} - c | < \epsilon$. For large x, $\varphi(x, p, b)$ may be approximated by the function $cr^{x-1}g(p, b)$, where

$$g(p, b) = pr^{-b} + (1 - p)r^{-b}, \tag{27}$$

a strictly convex function of b for each fixed p. In fact, provided $x - b - 1 > x_\epsilon$,

$$| \varphi(x, p, b) - cr^{x-1}g(p, b)| < \epsilon r^{x-b-1}.$$

For fixed p, $g(p, b)$ has a unique minimum over nonnegative values of b at the point $b = b^0(p)$, analogous to (10),

$$b^0(p) = \begin{cases} 0 & \text{if } p \leqq \tfrac{1}{2}, \\ \dfrac{\log \dfrac{1-p}{p}}{\log r^2} & \text{if } p > \tfrac{1}{2}, \end{cases}$$

the minimum value there being

$$g(p, b^0(p)) = \begin{cases} 1 & \text{if } \quad p \leqq \tfrac{1}{2}, \\ 2\sqrt{p(1-p)} & \text{if } \quad p > \tfrac{1}{2}. \end{cases}$$

Let δ be a small positive number and find a number p_δ, $\frac{1}{2} < p_\delta < 1$, so that F gives mass less than δ to the open interval $(p_\delta, 1)$. Let $\theta = b^0(p_\delta)$. Then, for $p < p_\delta$,

$$(28) \qquad \left| \inf_{0 \leqq b \leqq 2\theta} \varphi(x, p, b) - cr^{x-1} g(p, b^0(p)) \right| < \epsilon r^{x-2\theta-1},$$

and the infimum of $\varphi(x, p, b)$ is assumed at some point $b^*(x, p)$ for which

$$(29) \qquad | b^*(x, p) - b^0(p) | < \eta(\epsilon r^{-2\theta}),$$

where $\eta(\epsilon)$ is some function for which (i) $\eta(\epsilon) \to 0$ as $\epsilon \to 0$, and (ii) for b such that $| b - b^0(p)| > \eta(\epsilon)$, we have $g(p, b) > g(p, b^0(p)) + \epsilon$.

We will show that (28) remains valid for $p < p_\delta$ if the infimum is taken over the set $0 \leqq b \leqq x$. In fact, if $2\theta \leqq b \leqq x - x_\epsilon - 1$,

$$\begin{aligned} \varphi(x, p, b) &> r^{x-1}(cg(p, b) - \epsilon r^{-b}) \\ &\geqq r^{x-1}(cg(p, 2\theta) - \epsilon r^{-2\theta}) \geqq r^{x-1}(cg(p, b^0(p)) + \epsilon r^{-2\theta}), \end{aligned}$$

provided $\epsilon r^{-2\theta}$ is sufficiently small. Also, for values of b in the interval $x - x_\epsilon - 1 \leqq b \leqq x$,

$$\varphi(x, p, b) \geqq (1 - p)q(x_\epsilon) > r^{x-1}(cg(p, b^0(p)) + \epsilon r^{-2\theta}),$$

provided x is sufficiently large (with ϵ fixed). This proves for $p < p_\delta$, δ fixed, that

$$\left| \inf_{0 \leqq b \leqq x} \varphi(x, p, b) - cr^{x-1} g(p, b^0(p)) \right| < \epsilon r^{x-2\theta-1}$$

when ϵ is sufficiently small and x sufficiently large (depending on ϵ), and that the infimum value of $\varphi(x, p, b)$ is achieved at a point $b^*(x, p)$ which satisfies (29). In particular, letting $x \to \infty$ and then $\epsilon \to 0$ proves that

$$b^*(x, p) \to b^0(p) \quad \text{as} \quad x \to \infty$$

for all $p < p_\delta$. Since δ is arbitrary and since $b^*(x, 1) = x \to \infty = b^0(1)$, this is valid for all $p \leqq 1$.

In order to evaluate r, note that

$$\begin{aligned}
&\left| cr^x - cr^{x-1} \int_0^1 g(p, b^0(p))\, dF(p) \right| \\
&\leqq | cr^x - q(x) | + \left| q(x) - cr^{x-1} \int_0^1 g(p, b^0(p))\, dF(p) \right| \\
&\leqq \epsilon\, r^x + \int_0^{p_\delta} | \inf_{0 \leqq b \leqq x} \varphi(x, p, b) - cr^{x-1}\, g(p, b^0(p)) \,|\, dF(p) \\
&\qquad + \int_{p_\delta}^1 (\varphi(x, p, 0) + cr^{x-1}\, g(p, b^0\,(p)))\, dF(p) \\
&\leqq r^{x-1}(\epsilon r + \epsilon r^{-2\theta} + \delta(2c + \epsilon))
\end{aligned}$$

for x sufficiently large. Hence,

$$\left| r - \int_0^1 g(p, b^0(p))\, dF(p) \right| \leqq c^{-1}(\epsilon r + \epsilon r^{-2\theta} + \delta(2c + \epsilon)),$$

so that letting $\epsilon \to 0$ and then $\delta \to 0$ yields

$$\begin{aligned}
r &= \int_0^1 g(p, b^0(p))\, dF(p) \\
&= F(\tfrac{1}{2}) + 2 \int_{\frac{1}{2}}^1 \sqrt{p(1 - p)}\, dF(p),
\end{aligned} \tag{30}$$

exactly (11).

As an illustration of these results, consider the particular case when $F(p)$ is degenerate at some number $p_0 > \frac{1}{2}$. In such a case,

$$r = 2\sqrt{p_0(1 - p_0)},$$

$$b^0(p_0) = \frac{\log \dfrac{1 - p_0}{p_0}}{\log(4p_0(1 - p_0))}.$$

For example, when $p_0 = 2/3$, we find $r = .943 \cdots$ and $b^0(2/3) = \$5.89$. Thus, a rich man who pays a dollar a day to live, who receives each day an opportunity to make an even money wager on a gamble affording him probability $2/3$ of success, and who is interested only in survival, should bet approximately \$5.89 each day. It is interesting to note what happens for p_0 close to one-half or one: as $p_0 \to 1$, $b^0(p_0) \to 1$, and as $p_0 \to \frac{1}{2}$, $b_0^0(p_0) \to \infty$.

There are certain implications of a general nature describable in terms of individual behavior, which are worthwhile pointing out. In first studying these problems, the author was struck by the fact that in distinction with Kelly's betting system, the systems optimal for survival do not have bets which increase without bound as the resources increase without bound;

in other words, the optimal betting system for survival is quite conservative.

IMPLICATION 1. *If you have large resources and are interested only in survival, be conservative.*

The last sentence of the third paragraph above is also worth stating as a general implication.

IMPLICATION 2. *For rich individuals whose only interest is survival, those obtaining less favorable opportunities should be less conservative.*

It might be suspected that these results show that nobody is interested only in survival since nobody acts like this. The more resources a person gets, the more he sinks into the stock market. However, as a person's resources increase, so does his standard of living, and hence so will the bets he has to make to survive. To treat problems of this sort, the model would have to be modified.

A. J. Truelove has computed $b^*(x, p_0)$ in [6] for the above problem when $F(p)$ is degenerate at $p_0 = 2/3$. His results seem to bear out the conjecture that $b^*(x, p_0) \to b^0(p_0) = \$5.89$ as $x \to \infty$. One of the surprising results of his investigation is that for $x < x_0$, where x_0 is approximately \$9, $b^*(x, p_0) = x$. At x_0, $b^*(x, p_0)$ is discontinuous and drops to about \$4.50.

IMPLICATION 3. *If you are poor and interested only in survival, be bold.*

9. Model II. Instead of showing in detail how the conjecture (4) entails the asymptotic solution of the problem in Model II, we employ a more heuristic method to arrive at the solution.

Recall that Model II differs from the basic model only in that the betting restriction is

$$1 \leqq b_j \leqq x_{j-1}.$$

The Bellman functional equation corresponding to (25) for Model II problems is

$$q(x) = \int_0^1 \inf_{1 \leqq b \leqq x} \varphi(x, p, b)\, dF(p),$$

where

$$\varphi(x, p, b) = pq(x + b) + (1 - p)q(x - b)$$

represents the infimum probability of ruin given that $X_0 = x$, $p_1 = p$, and the bet b is made in game G_1. Again assuming q_x is continuous, there exists a function $b^*(x, p)$ satisfying $1 \leqq b^* \leqq x$, for which $\varphi(x, p, b^*) \equiv \inf_{1 \leqq b \leqq x} \varphi(x, p, b)$. And again we may conclude that the stationary Markov betting system which uses $b^*(x, p)$ at each stage is optimal.

Using the conjecture (4) that $q(x) \sim cr^x$, we find that $\varphi(x, p, b) \sim cr^x g(p, b)$, where $g(p, b)$ is defined by (27). The fundamental equation for large x is approximately

$$(31) \qquad 1 = \int_0^1 \inf_{1 \leq b} g(p, b)\, dF(p).$$

The infimum of $g(p, b)$ for $b \geqq 1$ is achieved at the point

$$b^0(p) = \max\left(1, \frac{\log \dfrac{1-p}{p}}{\log r^2}\right),$$

so that (31) is

$$1 = \int_0^1 g(p, b^0(p))\, dF(p),$$

which may be used to evaluate r. Equivalently, one may solve for π, the point at which $1 = \log((1-\pi)/\pi)/\log r^2$. Thus,

$$(32) \qquad 1 = \int_0^\pi [pr + (1-p)r^{-1}]\, dF(p) + \int_\pi^1 2\sqrt{p(1-p)}\, dF(p),$$

where $r = \sqrt{(1-\pi)/\pi}$. The right side of this equation can be shown to be an increasing function of π, to take a value less than one at $\pi = \frac{1}{2}$, to tend to infinity as π tends to one, and — surprisingly—to be continuous (surprising, because neither of the terms on the right side of (32) need be continuous in π separately). Therefore, there exists a unique value of π between $\frac{1}{2}$ and 1 at which (32) is satisfied, and for this value π_0 of π, the optimal betting system is

$$b^0(p) = \begin{cases} 1 & \text{if } p < \pi_0, \\ \dfrac{\log((1-p)/p)}{\log((1-\pi_0)/\pi_0)} & \text{if } p \geqq \pi_0. \end{cases}$$

Certain special cases are of interest:

(i) If F gives mass one to a point $p_1 > \frac{1}{2}$, then one can show that $\pi_0 \geqq p_1$, so that $b^0(p_1) = 1$; the rich player should always bet 1, the minimum bet.
(ii) If F gives mass $\frac{1}{2}$ to each of $p_1 > \frac{1}{2}$ and $q_1 = 1 - p_1$, then one can show that $q_1 < p_0 < p_1$, so that (32) becomes

$$1 = \tfrac{1}{2}[q_1 r + (1-q_1)r^{-1}] + \sqrt{p_1(1-p_1)},$$

an equation which defines the value of r. The asymptotically optimal bet at p_1 is

$$b(p_1) = \frac{\log \dfrac{1-p_1}{p_1}}{\log r^2}.$$

This is an increasing function of p_1, with

$$b(p_1) \to 1 + \sqrt{2} = 2.414 \cdots \quad \text{as} \quad p_1 \to \tfrac{1}{2},$$
$$b(\tfrac{2}{3}) \equiv 2.455 \cdots,$$
$$b(p_1) \to \infty \quad \text{as} \quad p_1 \to 1.$$

(iii) The introductory example. If F gives mass .2 to $p_1 = .6$, and mass .8 to $q_1 = .4$, then $r = .9765 \cdots$ and $b(p_1) = \$8.52$.

10. Discounted survival. In the previous two sections, betting systems were judged on the basis of survival forever. In this section we note briefly the consequences of using discounted future survival as the criterion. Consider a fixed betting system $\underset{\sim}{b} = \{b_j\}$ and let $q_i = q_i(x, \underset{\sim}{b})$ be the probability of being ruined by the play of the ith game, using betting system $\underset{\sim}{b}$ when the initial fortune is $X_0 = x$. We suppose that for some number ρ, $0 < \rho < 1$, the present value to the gambler of playing the ith game is ρ^{i-1}. Then, the present value of a history which is ruined by the ith game is $1 + \rho + \cdots + \rho^{i-1} = (1 - \rho^i)/(1 - \rho)$, and the present value of survival forever is $1/(1 - \rho)$. Hence, the value of a betting system $\underset{\sim}{b}$ when the initial fortune is $X_0 = x$ is

$$(1 - \rho)^{-1} \sum_{i=1}^{\infty} q_i(x, \underset{\sim}{b})(1 - \rho^i) + (1 - \rho)^{-1}(1 - \sum_{i=1}^{\infty} q_i(x, \underset{\sim}{b}))$$
$$= (1 - \rho)^{-1}(1 - G(\rho; x, \underset{\sim}{b})), \tag{33}$$

where G is the probability generating function of the time T at which ruin takes place

$$G(\rho; x, \underset{\sim}{b}) = E(\rho^T \mid X_0 = x, \underset{\sim}{b}) = \sum_{i=1}^{\infty} q_i(x, \underset{\sim}{b})\rho^i. \tag{34}$$

Rather than think of (33) as a class of utility functions, it is preferable to take (34) as a class of loss functions, since, when $\rho = 1$, G represents the probability of ruin, so that (34) may be considered as a generalization of the loss function used in the preceding sections.

It is valuable to consider two other interpretations of the criterion of discounted survival. Suppose that independent of what happens in the sequence of games being played, there is a fixed probability, $1 - \rho$, not depending on i, that between games G_i and G_{i+1} the gambler will separate from the system. Such a separation may be thought of as death, social revolution, etc. In other words, letting T^* represent the time at which separation takes place, we suppose that T^* and T are independent and that $P\{i < T^* < i + 1\} = (1 - \rho)\rho^i$. Since $P\{T < T^*\} = E\{P\{T < T^* \mid T\}\} = E\{\rho^t\}$, G represents the probability that the gambler is ruined before he separates from the system. In choosing $\underline{b}$ to minimize $G(\rho; x, \underline{b})$, we are thus minimizing the probability of being ruined before separation. On the other hand, if we consider separation and ruin both as death, and suppose that $P\{T^* = i + 1\} = (1 - \rho)\rho^i$ with T^* independent of T, then since

$$E\{\min (T, T^*)\} = E\{E\{\min (T, T^*)| T\}\}$$
$$= E\left\{\sum_1^T i(1 - \rho)\rho^{i-1} + T \sum_{T+1}^{\infty} (1 - \rho)\rho^{i-1}\right\} = \frac{1 - E\rho^T}{1 - \rho},$$

we see that (33) represents the expected length of life. In choosing $\underline{b}$ to minimize $G(\rho; x, \underline{b})$, we are thus maximizing the expected length of life. It should be noted that both these interpretations, minimum probability of ruin before separation and maximum expected length of life, are valid even if the game is not favorable. The results of this section apply to nonfavorable games as well.

Let

$$q_\rho(x) = \inf_b G(\rho; x, \underline{b})$$

so that $q_1(x)$ is the $q(x)$ of (2). One can derive as before under the same general conjectures (that $q_\rho(x) \sim c_\rho r_\rho^{\,x}$ for some $0 < r_\rho < 1$ and $c_\rho > 0$, that attention may be restricted to stationary strategies, and that $q_\rho(x)$ is continuous in x) the asymptotic value of the optimal strategy.

First we consider Model I. The equation analogous to (26) is found to be

$$q_\rho(x) = \rho \int_0^1 \inf_{0 \leqq b \leqq x} [pq_\rho(x + b - 1) + (1 - p)q_\rho(x - b - 1)]\, dF(p),$$

Boldly substituting $c_\rho r_\rho^{\,x}$ for $q_\rho(x)$ in this equation, we find the asymptotically optimal betting system to be

$$b_\rho(p) = \begin{cases} 0 & \text{if } p \leqq \tfrac{1}{2}, \\ \dfrac{\log \dfrac{1-p}{p}}{\log r_\rho^{\,2}} & \text{if } p > \tfrac{1}{2}, \end{cases}$$

where r_ρ this time satisfies the equation, analogous to (30),

$$r_\rho = \rho F(\tfrac{1}{2}) + 2\rho \int_{\frac{1}{2}}^1 \sqrt{p(1 - p)}\, dF(p).$$

Since $r_\rho = \rho r_1$ and $b_\rho(p) = b_1(p) \log r_1/(\log \rho + \log r_1)$, we see that the asymptotically optimal betting system in the discounted case is a constant proportion of the betting system in the nondiscounted case. As an example, suppose that F is degenerate at $p_0 = 2/3$, and that $\rho = .9$; then $r_\rho = .849$ and $b_\rho(p_0) = \$2.11$. It is reasonable that this should be a smaller bet than the

\$5.89 which is asymptotically optimal when $\rho = 1$, because when $\rho = .9$ survival in the distant future is not as important to the bettor. The more discounted the future, the more conservative the betting system.

The results for Model II are completely analogous. The fundamental equation is

$$q_\rho(x) = \rho \int_0^1 \inf_{1 \leq b \leq x} [p\, q_\rho(x+b) + (1-p)\, q_\rho(x-b)]\, dF(p)$$

and the asymptotically optimal betting system is

$$b_\rho(p) = \max\left(1, \frac{\log \dfrac{1-p}{p}}{\log r_\rho^2}\right),$$

where r_ρ satisfies the equation analogous to (32),

$$\rho^{-1} = \int_0^{\pi_\rho} [pr_\rho + (1-p)r_\rho^{-1}]\, dF(p) + \int_{\pi_\rho}^1 2\sqrt{p(1-p)}\, dF(p), \tag{35}$$

with $r_\rho^2 = (1-\pi_\rho)/\pi_\rho$. The statements used in §9 to show that a root of (32) exists, may also be used to show that a root of (35) exists, and that r_ρ is a decreasing function of ρ. As a numerical illustration, consider the introductory example with $\rho = .9$. For this value of ρ, $r_\rho = .695 \cdots$, and the asymptotically optimal betting system is to bet \$1 regardless of the value of p. It is interesting to note that such a betting system does not give a positive probability of survival.

We remark in conclusion that the results of these last three sections can be generalized to many other models, including the mixture of Models I and II wherein there is a minimum bet and a cost of living, and including models wherein there is also a maximum bet.

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