THE ENDGAME IN POKER

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Abstract. The simple two-person poker model, known as Basic Endgame, may be described as follows. With a certain probability known to both players, Player I is dealt a hand which is a sure winner. If it is not a sure winner, then Player II is sure to have a better hand. Player I may either check, in which case the better hand wins the antes, or bet. If Player I bets, Player II may either fold, conceding the antes to Player I, or call, in which case the better hand wins the antes plus the bets. This model is reviewed and extended in several ways. First, several rounds of betting are allowed before the hands are compared. Optimal choice of the sizes of the bets are described. Second, Player II may be given some hidden information concerning the probability that Player I has a sure winner. This information is given through the cards Player II receives. Third, the model is extended to the case where the hand Player I receives only indicates the probability it is a winner. This allows for situations in which cards still to be dealt may influence the outcome.

Introduction.

In this investigation, we treat from a game-theoretic point of view several situations that occur in the game of poker. The analysis of models of poker has a long and distinguished existence in the game theory literature. Chapter 5 of the book of Émile Borel, *Applications aux jeux de hasard* (1938), and Chapter 19 of the seminal book on game theory by von Neumann and Morgenstern (1944) are devoted to the topic. In the 1950's, others developed further certain aspects of modeling poker. Kuhn (1950) treats three card poker. Nash and Shapley (1950) treat a three person poker model. Bellman and Blackwell (1949), Bellman (1952), Gillies, Mayberry and von Neumann (1953), Karlin and Restrepo (1957), Goldman and Stone (1960ab), and Pruitt (1961) extend various aspects of the poker models of Borel and of von Neumann-Morgenstern further. Chapter 9 of the textbook of Karlin (1959) summarizes this development. For a more recent treatment of the models of Borel and von Neumann, see Ferguson and Ferguson (2003) and Ferguson, Ferguson and Gawargy (2007).

Generally, the aim of such research is to analyze a simplified model of the game of poker completely, with the hope of capturing the spirit of poker in a general sense. Others have tried to analyze specific situations or aspects of the real game with the hope that it may improve one's play. Papers of Newman (1959), Friedman (1971), Cutler (1975, 1976), and Zadeh (1977), and the book of Ankeny (1981) are of this category. On the other hand there are also books that contain valuable information and recommendations on how to play the real game of poker. The books of Brunson (1978) and Sklansky (1987) on general games of poker and of Sklansky and Malmuth (1988), Sklansky, Malmuth and Zee (1989), Zee (1992) and Harrington and Robertie (2004-2006) on specific games of poker can be recommended. Unusual in its treatment mixing mathematical analysis, game theoretic ideas and the real game of poker, the book of Chen and Ankenman (2006) may be especially recommended.

One of the simplest and most useful mathematical models of a situation that occurs in poker is called the "classical betting situation" by Friedman (1971) and "basic endgame" by Cutler (1976). These papers provide explicit situations in the game of stud poker and of lowball stud for which the model gives a very accurate description. This model is also found in the exercises of the book of Ferguson (1967). Since this is a model of a situation that occasionally arises in the last round of betting when there are two players left, we adopt the terminology of Cutler and call it Basic Endgame in poker. This will also emphasize what we feel is an important feature in the game of poker, that like chess, go, backgammon and other games, there is a distinctive phase of the game that occurs at the close, where special strategies and tactics that are analytically tractable become important.

1. Basic Endgame.

Basic Endgame is played as follows. Two players, Player I and Player II, both put an ante of a dollars into a pot ($a > 0$). Player I then draws a card from a deck of cards that gives him a winning card with probability (w.p.) P and a losing card w.p. $1 - P$, $0 < P < 1$. Both players know the value of P, but only Player I knows if the card he received is a winning card or not. Player I may then check (also called pass) or bet b dollars ($b > 0$). If Player I checks, the game is over and the antes goes to Player I if he has a winning card and to Player II otherwise. If Player I bets, Player II may then fold, or she may call by also putting b dollars in the pot. If she folds, then Player I wins the ante whatever card he has. If Player II calls, then the ante plus the bet is won by Player I if he has a winning card and by Player II otherwise. Only the ratio of b to a is significant, but we retain separate symbols for the ante and the bet so that the results are easier to understand.

Situations of the form of Basic Endgame arise in poker. For example, in the last round of betting in a game of five card stud poker, Player I's cards are the 5 of diamonds, the 6 of spades, the 7 of diamonds, the 8 of hearts and a hidden hole card. Player II's cards are the 2 of hearts, the 3 of spades, the king of spades, the king of clubs and a hidden hole card. No matter what card Player II has in the hole, Player I will win if and only if he has a 4 or a 9 in the hole. Since Player II has the higher hand showing, she must act first by betting or checking. In this situation, it is optimal for her to check. Assuming she does check, it then becomes Player I's turn to act, and we have a situation close to Basic Endgame described in the previous paragraph. The number a may be taken to be half the size of the present pot, and b will be the maximum allowable bet. The number P is taken to be the probability that Player I has a winning hole card given the past history of the game.

There are several reasons why Basic Endgame is not a completely accurate description of this poker situation. In the first place, the probability P, calculated on the basis of three rounds of betting before the final round and the actions of the players in these rounds, is an extraordinarily complex entity. It would be truly remarkable if both players arrived at the same evaluation of P as required by Basic Endgame. Secondly, Player II's hole card gives her some secret information unknown to Player I that influences her estimation of P, and both Player I and Player II must take this into account. In Section 5, we extend the model to allow for this hidden type of information. Thirdly, a player may unknowingly give away information through mannerisms, hesitations, nervousness, etc. This type of hidden information, called "tells", (see Caro (1984)) is of a different category than information given by a hidden hole card. The player who gives away such information has no way of taking this into account since he is unaware of its existence. Another way this type of hidden information can arise is through cheating. For example, someone may gather information about the cards through a hole in the ceiling and pass this information to one of the players at the table. (Don't laugh—this happened at one of the casinos in California.) Such games in which one of the players does not know all the rules of the game, are called pseudo-games, and have been studied by Baños (1968) and Megiddo (1980). The extensive literature on repeated games of incomplete information (e.g. see Aumann and Mashler (1995) or Sorin (2000)) is also an attempt to treat this problem.

If Player I receives a winning card, it is clear that he should bet: If he checks, he wins a net total of a dollars, whereas if he bets he will win at least a dollars and possibly more. In the analysis of the game below, we assume that Player I will bet with a winning card.

The rules of the game may be summarized in a diagram called the Kuhn tree, a device due to Kuhn (1953). Figure 1 gives the Kuhn tree of Basic Endgame. It is to be read from the top down. The first move is a chance move with probabilities P and $1-P$ attached to the edges. Then Player I moves, followed by Player II. The payoffs to Player I are attached to each terminal branch of the tree. The only features not self-explanatory are the circle and the long oval. These represent information sets. The player whose turn it is to move from such a set does not know which node of the set the previous play has led to. Thus the long oval indicates that Player II does not know which of the two nodes she is at when she makes her choice, whereas the circle indicates that Player I does learn the outcome of the chance move.

In this situation, Player I has two possible pure strategies: (a) *the bluff strategy*—bet with a winning card or a losing card; and (b) *the honest strategy*—bet with a winning card and check with a losing card. Player II also has two pure strategies which are (a) *the call strategy*—if Player I bets, call; and (b) *the fold strategy*—if Player I bets, fold. The payoff matrix is the two by two matrix of expected winnings of Player I,

$$\begin{array}{c c} & \begin{array}{cc} \textit{fold} & \textit{call} \end{array} \\ \begin{array}{c} \textit{honest} \\ \textit{bluff} \end{array} & \begin{pmatrix} (2P-1)a & (2P-1)a+Pb \\ a & (2P-1)(a+b) \end{pmatrix} \end{array}$$

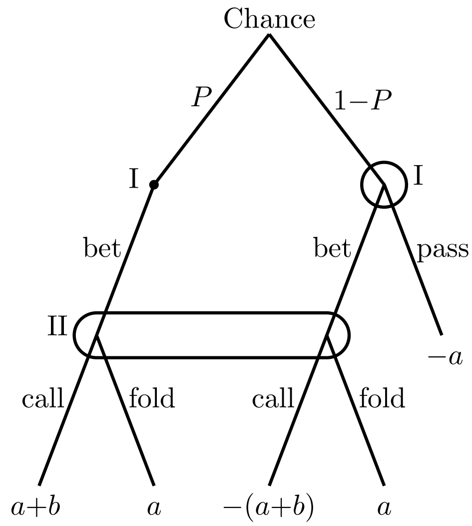

Figure 1.

We state the solution to Basic Endgame by giving the value and optimal (i.e. minimax) strategies for the players. There are two cases. In the first case, for values of P close to one, there is a saddle-point — the players have optimal pure strategies. The main case is for small values of P, when both players use both strategies with positive probabilities. This is called the all-strategies-active case.

If $P \geq (2a+b)/(2a+2b)$, then there is a saddlepoint.
(i) I's optimal strategy is to bet always.
(ii) II's optimal strategy is to fold always.
(iii) The value of the game is a.

If $P < (2a+b)/(2a+2b)$, then we are in the all-strategies-active case.
(i) I's optimal strategy is to bluff w.p. $\pi := (b/(2a+b))(P/(1-P))$.
(ii) II's optimal strategy is to fold w.p. $\phi := b/(2a+b)$.
(iii) The value of the game is $2aP[(2a+2b)/(2a+b)] - a$.

These strategies have an easy derivation and interpretation using one of the basic principles of game theory called *The Indifference Principle: In those cases where your opponent, using an optimal strategy, is mixing certain pure strategies against you, play to make your opponent indifferent to which of those strategies he uses.*

In Basic Endgame, this principle may be used as follows. In the all-strategies-active case, II plays to make I indifferent to betting or checking with a losing card. If I passes, he wins $-a$. If he bets, he wins a w.p. ϕ and wins $-(a+b)$ w.p. $1-\phi$, where ϕ is the (unknown) probability that II folds. His expected return in this case is $\phi a - (1-\phi)(a+b)$. Player I is indifferent if $\phi a - (1-\phi)(a+b) = -a$. This occurs if $\phi = b/(2a+b)$. Therefore, II should choose to fold w.p. $b/(2a+b)$.

Similarly, I chooses the probability π of betting with a losing card to make II indifferent to calling or folding. If II calls, she loses $(a+b)$ w.p. P, while w.p. $1-P$ she wins $(a+b)$ w.p. π and wins a w.p. $1-\pi$; her expected loss is therefore, $P(a+b)-(1-P)\pi(a+b)-(1-P)(1-\pi)a$. If she folds, she loses a w.p. P, while w.p. $1-P$, she loses a w.p. π and wins a w.p.

$1-\pi$; in this case her expected loss is $a[P+(1-P)\pi-(1-P)(1-\pi)]$. She is indifferent between calling and folding if these are equal, namely if $\pi = (P/(1-P))(b/(2a+b))$. Therefore, I should bluff with this probability.

It is interesting that Player II's optimal strategy does not depend on P in the all-strategies-active case. In particular in pot limit poker where $b = 2a$, her optimal strategy is to call w.p. 1/2 and fold w.p. 1/2. However, Player II must check the condition for the all-strategies-active case before using this strategy. This condition is automatic for I. If I computes π and it is greater than 1, then he should bluff w.p. one.

Choosing the Size of the Bet.

In Basic Endgame, the size of the bet of Player I was fixed at some number $b > 0$. The general situation where I is allowed to choose any positive bet size, b, less that some maximum amount, B, was investigated in an unpublished paper by Cutler (1976). The conclusion is that Player I may as well always bet the maximum; in fact, in the all-strategies-active case it is a mistake for I to bet less than the maximum. (By a *mistake* for Player I, we mean that Player II can take advantage of such a bet without risk and achieve a strictly better result than she could against optimal play of the opponent.) It is dangerous for I to let the size of the bet depend on whether he has a winning card or not. II may be able to take advantage of this information. Also, if I bets the same amount regardless of his hand, he might as well bet the maximum since the value to him is a nondecreasing function of the bet size.

Cutler gives optimal strategies for Player II which allow her to take advantage of any variation in I's bet sizes without incurring any risk. His general result is as follows, where B represents the maximum allowable bet size.

If $P \geq B/(2a+B)$, Player I should always bluff with a losing hand. Player II should always fold no matter how big or small I's bet is.

If $P < B/(2a+B)$, Player I should bluff by betting B w.p. $(B/(2a+B))(P/(1-P))$. If Player II hears a bet of size b, $0 < b \leq B$, then she should call w.p. $p(b)$ where

$$\frac{2a}{2a+b} \leq p(b) \leq \min\{1, \frac{2aB}{(2a+B)b}\}.$$

Any such $p(b)$ is minimax for Player II. In particular, she may pretend that the bet size was fixed at b and use the solution to Basic Endgame, namely call w.p. $2a/(2a+b)$. When $b < B$, this gives her an improved expected payoff against all strategies of Player I except strategies that only bet less than B when I has a losing card. To obtain an improved expected payoff against all strategies, she should call a little more often, but with probability still less than $2aB/((2a+B)b)$.

2. Basic Endgame with Two Rounds of Betting.

It sometimes happens that Player I will face an endgame situation with two or more rounds of betting yet to take place, in which the cards to be dealt between rounds do not affect the outcome. For two rounds, this is modelled as follows.

Two players both ante a units into the pot. Then Player I receives a winning card w.p. $P \geq 0$ and a losing card w.p. $1 - P \geq 0$. It is assumed that I knows which card he has whenever he makes a decision, and II does not know which card I holds whenever she makes a decision. Player I then either passes or bets an amount $b_1 > 0$. If he passes, the game is over and he wins a if he holds the winning card and loses that amount if he holds the losing card. If I bets, Player II may call or fold. If II folds the game is over and I wins a. If II calls, I may either pass or bet $b_2 > 0$. If he passes, the game is over and he wins $a + b_1$ if he holds the winning card and loses that amount if he holds the losing card. If he bets, then II may call or fold. If II folds, then the game is over and I wins $a + b_1$. If II calls, then the game is over and I wins $a + b_1 + b_2$ if he holds the winning card and loses that amount if he holds the losing card.

If I receives a winning card, it is clear he should never pass. We assume the rules of the game require him to bet in this situation. With such a stipulation, the game tree is displayed in Figure 2.

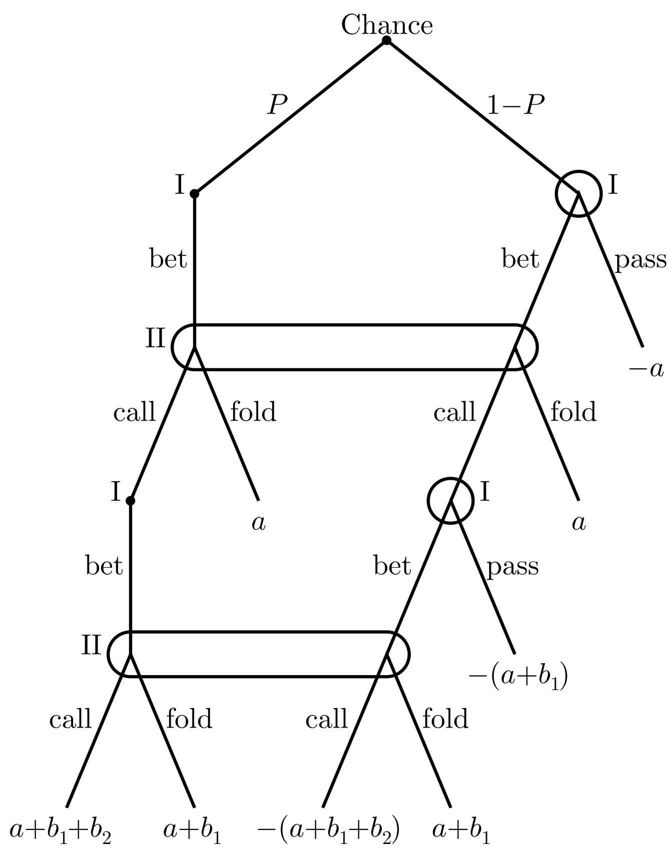

Figure 2.

If I chooses to pass at the first round, then it does not matter what he does in the second round. So I has just three pure strategies, pass, bet-pass, and bet-bet. Similarly, II has just three strategies, fold, call-fold and call-call. The resulting three by three matrix of expected payoffs is

$$
\begin{array}{cccc}
 & \text{fold} & \text{call-fold} & \text{call-call} \\
\begin{array}{l}\text{pass}\\ \text{bet-pass}\\ \text{bet-bet}\end{array} & \multicolumn{3}{c}{\begin{pmatrix} a(2P-1) & a(2P-1)+Pb_1 & a(2P-1)+P(b_1+b_2) \\ a & (2P-1)(a+b_1) & (2P-1)(a+b_1)+Pb_2 \\ a & a+b_1 & (2P-1)(a+b_1+b_2) \end{pmatrix}}
\end{array}
$$

We state the solution to this game. Since this is a special case of the problem treated in the next section, we omit the proof. Let

$$P_0 := \frac{(2a+b_1)(2a+2b_1+b_2)}{(2a+2b_1)(2a+2b_1+2b_2)}.$$

If $P > P_0$, then
(i) the value is $V = a$,
(ii) it is optimal for Player II to fold on the first round, and
(iii) it is optimal for Player I to bet on the first round, and to bet w.p. $(P/(1-P))(b_2/(2a+2b_1+b_2))$ (or w.p. 1 if this is greater than 1) on the second round.

If $P \leq P_0$, then all strategies are active,
(i) the value is $V = a(2P - P_0)/P_0$
(ii) it is optimal for Player II to fold on the first round w.p. $b_1/(2a+b_1)$, and to fold on the second round w.p. $b_2/(2a+2b_1+b_2)$, and
(iii) with a winning card, Player I always bets; with a losing card, he bets on the first round w.p. $\frac{P}{1-P}\cdot\frac{1-P_0}{P_0}$, and on the second round w.p. $\frac{b_2(2a+b_1)}{b_2(2a+b_1)+2b_1(a+b_1+b_2)}$.

Note the following features. The cutoff-point, P_0, between the two cases is just the product of the cutoff points of the two rounds treated separately, that is $(2a+b_1)/(2a+2b_1)$ for the first round and $(2a+2b_1+b_2)/(2a+2b_1+2b_2)$ for the second round. The first case, $P > P_0$, occurs if and only if the lower right 2 by 2 submatrix of the payoff matrix has value at least a. If Player I uses the optimal strategy for this 2 by 2 submatrix, then his expected payoff is at least a, and since he can get no more than a if Player II always folds, the value must be a. In this sense, the first case is easy.

In the all-strategies-active case where $P \leq P_0$, Player II's optimal strategy is just the strategy that uses her optimal strategy for Basic Endgame in both the first and second round. In the first round, Player II sees a pot of size $2a+b_1$ and is required to invest b_1 to have a chance to win it. Therefore, she folds w.p. $b_1/(2a+b_1)$. In the second round, she sees a pot of size $2a+2b_1+b_2$ and is required to call with b_2 to continue. Therefore, she folds with conditional probability $b_2/(2a+2b_1+b_2)$.

Player I's optimal strategy in the all-strategies-active case makes Player II indifferent to folding or calling in both the first and the second round. It is interesting to note that Player I's behavioral strategy in the second round is independent of P.

Choosing the Sizes of the Bets.

In choosing the sizes of the bets in the all-strategies-active case, three cases deserve special mention. The first case is the case of limit poker, where there is a fixed upper limit, B, on the size of every bet. The value, V, is an increasing function of both b_1 and b_2, so the optimal choice of bet size for Player I is $b_1 = b_2 = B$. The formulas for the optimal strategies and the value do not simplify significantly in this case.

The second case is the pot-limit case. Since V is increasing in both b_1 and b_2, the optimal choices are $b_1 = 2a$ and $b_2 = 6a$, the pot-limit bets. The basic formulas for the solution simplify in this case. The inequality defining the all-strategies-active case reduces in this case to $P \leq 4/9$. The value is $V = (9P - 2)a/2$. The optimal strategy of Player II is: *Call w.p.* $1/2$ *on both the first and second rounds.* The optimal strategy for Player I is: *Bet with a winning card; with a losing card bet on the first round w.p.* $(5/4)(P/(1-P))$*, and on the second round w.p.* $2/5$.

The third case is no-limit (table-stakes) poker, in which a player may bet as much as he likes but no more than he has placed on the table when play begins. In addition, if the amount bet exceeds that amount he has left, he may call that part of the bet up to the amount he has left. If there are two players with remaining resources, B_1 and B_2, the maximum bet size is for all practical purposes $B = \min\{B_1, B_2\}$. As in Section 1, it is optimal for Player I to bet the maximum eventually, but the question remains of how much of it to bet on the first round.

Suppose therefore that the sum of the bets, $b_1 + b_2 = B$, is fixed, and Player I is allowed to choose the size of the first bet, b_1, subject to $0 \leq b_1 \leq B$. Then in the all-strategies-active case, the optimal value of b_1 is to maximize

$$V = a\left(2P\frac{(2a+2b_1)(2a+2b_1+2b_2)}{(2a+b_1)(2a+2b_1+b_2)} - 1\right) = a\left(2P\frac{(2a+2b_1)(2a+2B)}{(2a+b_1)(2a+b_1+B)} - 1\right).$$

The value of b_1 that maximizes V is easily found by setting the derivative of V with respect to b_1 to zero. Solving the resulting equation reveals the optimal choice of b_1 to be

$$b_1 = \sqrt{aB + a^2} - a.$$

3. Basic Endgame With Many Rounds of Betting.

We may extend Basic Endgame to allow an arbitrary finite number, n, of betting rounds.

Two players both ante a units into the pot, $a > 0$. Then Player I receives a winning card w.p. $P \geq 0$ and a losing card w.p. $1 - P \geq 0$. It is assumed that I knows which card he has whenever he makes a decision, and II does not know which card I holds whenever she makes a decision. Player I then either passes or bets an amount $b_1 > 0$. If he passes, the game is over and he wins a if he holds the winning card and loses that amount if he holds the losing card. If I bets, Player II may call or fold. If II folds the game is over and

I wins a. If II calls, the game enters round 2. In round k where $2 \le k < n$, I may either pass or bet $b_k > 0$. If he passes, the game is over and he wins $a + b_1 + \cdots + b_{k-1}$ if he holds the winning card and loses that amount if he holds the losing card. If he bets, then II may call or fold. If II folds, then the game is over and I wins $a + b_1 + \cdots + b_{k-1}$. If II calls, then the game enters round $k+1$. If II calls in round n, the game is over and I wins $a + b_1 + \cdots + b_n$ if he holds the winning card and loses that amount if he holds the losing card. We first assume the b_k are fixed numbers.

Below, we derive the value optimal strategies of the players. We summarize the solution as follows.

Summary of Solution. *Let*

$$r_k := \frac{b_k}{2(a + b_1 + \cdots + b_{k-1}) + b_k}$$

and let $P_0 = \left(\prod_1^n (1 + r_j)\right)^{-1}$.

If $P > P_0$*, then it is optimal for Player I to bet on the first round and for Player II to fold. The value is* $V = a$.

If $P \le P_0$*, then*
(1) the value is $V = a(2P - P_0)/P_0$*,*
(2) Player II's optimal strategy is at each stage k *to fold w.p.* r_k*, and*
(3) Player I's optimal strategy is to bet with a winning card; with a losing card, to bet on the first stage w.p. $p_1 = \dfrac{P}{1-P} \cdot \dfrac{1 - P_0}{P_0}$*, and if stage* $k > 1$ *is reached, to bet w.p.* $p_k = \dfrac{\prod_k^n (1 + r_j) - 1}{\prod_{k-1}^n (1 + r_j) - 1}$.

We notice some remarkable features of this solution in the all-strategies-active case, $P \le \left(\prod_1^n (1 + r_j)\right)^{-1}$. First note that r_k is just the amount bet at stage k divided by the new pot size. This means that Player II's optimal strategy is just the repeated application of Player II's optimal strategy for Basic Endgame. In addition, Player I's optimal strategy depends on P only at the first stage; thereafter his behavior is independent of P. After the first stage, he will bet with the same probabilities if P is very small, say $P = .001$, as he would if $P = .1$. This means that his main decision to bluff is taken at the first stage; thereafter all bluffs are carried through identically.

Derivation. If I receives a winning card, it is clear he should never pass. We assume the rules of the game require him to bet in this situation.

Player I has $n+1$ pure strategies, $i = 0, 1, \ldots, n$, where i represents the strategy that bluffs exactly i times. Similarly, there are $n+1$ pure strategies for Player II, $j = 0, 1, \ldots, n$, where j represents the strategy that calls exactly j times. Let A_{ij} denote the expected payoff to I if I uses i and II uses j. Let

$$s_j := a + b_1 + \cdots + b_j$$

denote half the size of the pot after round j; in particular, $s_0 = a$. Then

$$A_{ij} = \begin{cases} Ps_j - (1-P)s_i & \text{for } 0 \le i \le j \le n \\ s_j & \text{for } 0 \le j < i \le n. \end{cases} \tag{1}$$

Let $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ denote the mixed strategy for Player II in which σ_j is the probability that II calls exactly j times. If II uses this strategy and Player I uses i, the average payoff is

$$\begin{aligned} V_i := \sum_{j=0}^{n} A_{ij}\sigma_j &= \sum_{j=0}^{i-1} s_j\sigma_j + P\sum_{j=i}^{n} s_j\sigma_j - (1-P)s_i\sum_{j=i}^{n}\sigma_j \\ &= P\sum_{j=0}^{n} s_j\sigma_j + (1-P)\sum_{j=0}^{i-1} s_j\sigma_j - (1-P)s_i\sum_{j=i}^{n}\sigma_j. \end{aligned} \tag{2}$$

We search for a strategy $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ to make V_i independent of i. Such a strategy would guarantee that Player II's average loss would be no more than the common value of the V_i. We look at the differences

$$V_k - V_{k-1} = (1-P)[(s_k + s_{k-1})\sigma_{k-1} - (s_k - s_{k-1})\sum_{j=k-1}^{n}\sigma_j] \tag{3}$$

This is zero for $k = 1, \ldots, n$ if

$$\frac{\sigma_{k-1}}{\sum_{j=k-1}^{n}\sigma_j} = \frac{s_k - s_{k-1}}{s_k + s_{k-1}} = r_k \tag{4}$$

This defines the σ_k. In fact, the left side represents the probability II folds in round k given that it has been reached, and so is the behavioral strategy for II in round k. The equalizing value of the game may be found as follows.

$$\begin{aligned} V_0 &= P\sum_{j=0}^{n} s_j\sigma_j - (1-P)s_0 \\ V_n &= \sum_{j=0}^{n-1} s_j\sigma_j + Ps_n\sigma_n - (1-P)s_n\sigma_n = \sum_{j=0}^{n} s_j\sigma_j - 2(1-P)s_n\sigma_n \end{aligned} \tag{5}$$

From $V_0 = V_n$, we see that $\sum_{j=0}^{n} s_j\sigma_j = 2s_n\sigma_n - s_0$. This gives the value as $V_0 = 2Ps_n\sigma_n - s_0$. Finally, repeatedly using (4) in the form $1 - r_k = \sum_k^n \sigma_j / \sum_{k-1}^n \sigma_j$, we find that $\prod_{i=1}^{n}(1-r_i) = \sigma_n$. Hence the value is

$$V_0 = 2Ps_n\prod_{i=1}^{n}(1-r_i) - s_0.$$

Noting that $1 - r_k = 2s_{k-1}/(s_k + s_{k-1})$ and $1 + r_k = 2s_k/(s_k + s_{k-1})$, we find that $s_n \prod_1^n (1 - r_k) = s_0 \prod_1^n (1 + r_k)$. This, with $s_0 = a$, gives an alternate form of V_0, namely

$$V_0 = a\left(2P\prod_1^n (1 + r_k) - 1\right) = a\left(2\frac{P}{P_0} - 1\right). \tag{6}$$

Player II can keep the value of the game to be at most V_0. But Player II can also keep the value to be at most a by folding always. We shall now see by examining Player I's strategies that the value of the game is the minimum of (6) and a.

Let $(\pi_0, \pi_1, \ldots, \pi_n)$ denote a mixed strategy of Player I, where π_i is the probability of making exactly i bets. If Player I uses this strategy and Player II uses column j, the average payoff is for $0 \le j \le n$,

$$\begin{aligned} W_j := \sum_{i=0}^n \pi_i A_{ij} &= \sum_{i=0}^j \pi_i(Ps_j - (1-P)s_i) + \sum_{i=j+1}^n \pi_i s_j \\ &= Ps_j + (1-P)s_j \sum_{i=j+1}^n \pi_i - (1-P)\sum_{i=0}^j s_i \pi_i. \end{aligned} \tag{7}$$

Equating W_j and W_{j-1} leads to the following simultaneous equations for $1 \le j \le n$,

$$P(s_j - s_{j-1}) + (1-P)(s_j - s_{j-1}) \sum_{i=j+1}^n \pi_i - (1-P)(s_j + s_{j-1})\pi_j = 0 \tag{8}$$

Solving for π_j yields the equations,

$$\pi_j = \frac{P}{1-P} r_j + r_j \sum_{i=j+1}^n \pi_i. \tag{9}$$

which defines $\pi_n, \ldots, \pi_1$ by backward induction. We find $\pi_n = (P/(1-P))r_n$, and

$$\sum_{i=j+1}^n \pi_i = \frac{P}{1-P}\left[\prod_{i=j+1}^n (1 + r_i) - 1\right], \tag{10}$$

so that Player I's behavioral strategy at stages $2 \le j \le n$ is

$$p_j = \frac{\sum_j^n \pi_i}{\sum_{j-1}^n \pi_i} = \frac{\prod_j^n (1 + r_i) - 1}{\prod_{j-1}^n (1 + r_i) - 1}. \tag{11}$$

The behavioral strategy at the first stage is

$$p_1 = 1 - \pi_0 = \sum_1^n \pi_i = \frac{P}{1-P}\left[\prod_1^n (1 + r_i) - 1\right]. \tag{12}$$

Assuming the resulting $p_1 \leq 1$, we can evaluate the common value of the W_j using

$$W_0 = Ps_0 + (1-P)s_0 \sum_{i=1}^{n} \pi_i - (1-P)s_0\pi_0 = 2Ps_0 \prod_{1}^{n}(1+r_i) - s_0 = V_0. \qquad (13)$$

Therefore, $W_0 = V_0$ is the value of the game provided $p_1 \leq 1$, or equivalently, provided $P\prod_1^n(1+r_i) \leq 1$.

Finally it is easily checked that $p_1 \leq 1$ if and only if V_0 is not greater than s_0.

Choosing the sizes of the bets.

Suppose that the initial size of the pot, $2b_0 = 2s_0$, is fixed, and that the total amount to be bet, $b_1 + \cdots + b_n = s_n - s_0$, is fixed, where n is the number of rounds of betting. This situation occurs in no-limit, table stakes games, where $s_n - s_0$ is the minimum of the stakes that Player I and Player II have in front of them when betting begins. In the last round, Player I should certainly bet the maximum amount possible. The problem for Player I is to decide how much of the total stakes to wager on each intervening round. This is equivalent to finding the choices of $s_1, s_2, \ldots, s_{n-1}$ that maximize the value, V_0, of (6) subject to the constraints,

$$s_0 \leq s_1 \leq s_2 \leq \cdots \leq s_n. \qquad (14)$$

Maximizing V_0 is equivalent to maximizing

$$\prod_{i=1}^{n}(1+r_i) = \prod_{i=1}^{n} \frac{2s_i}{s_i + s_{i-1}}. \qquad (15)$$

As a function of s_i for fixed $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n-1}$, this proportional to

$$\frac{s_i}{(s_i + s_{i-1})(s_{i+1} + s_i)} \qquad (16)$$

This is unimodal in s_i on $(0, \infty)$ with a maximum at $(s_i + s_{i-1})(s_{i+1} + s_i) = s_i[s_{i+1} + 2s_i + s_{i-1}]$, or equivalently, at

$$s_i = \sqrt{s_{i-1}s_{i+1}} \qquad (17)$$

Note that s_i is the geometric mean of s_{i-1} and s_{i+1} so that s_i is between s_{i-1} and s_{i+1}. Thus, the global maximum of (15) subject to (14) occurs when (17) is satisfied for all $i = 1, 2, \ldots, n-1$. Inductively using (17) from $i = 1$ to $n-1$, we can find s_i in terms of s_0 and s_{i+1} to be $s_i = s_0^{1/(i+1)} s_{i+1}^{i/(i+1)}$ for $i = 1, \ldots, n$. For $i = n-1$, this is $s_{n-1} = s_0^{1/n} s_n^{(n-1)/n}$. We may now work back to find

$$s_i = s_0^{(n-i)/n} s_n^{i/n} = s_0 (s_n/s_0)^{i/n} \qquad \text{for } i = 1, \ldots, n-1. \qquad (18)$$

From this we may find the folding probabilities for Player II.

$$r_i = \frac{s_n^{1/n} - s_0^{1/n}}{s_n^{1/n} + s_0^{1/n}} \qquad \text{for } i = 1, \ldots, n-1. \tag{19}$$

Note that this is independent of i. Since this is the ratio of bet size to the size of the new pot, one sees that the optimal bet size must be a fixed proportion of the size of the pot! This proportion is easily computed to be

$$\frac{r}{1-r} = \frac{(s_n/s_0)^{1/n} - 1}{2}, \tag{20}$$

where r denotes the common value of the r_i of (19). Note that this is independent of P! Therefore it is optimal for Player I to bet this proportion of the pot at each stage.

Let us compute Player I's optimal bluffing probabilities. At the initial stage, I should bet (with a losing card) w.p.

$$p_1 = \frac{P}{1-P}[(r+1)^n - 1], \tag{21}$$

If $p_1 > 1$, then Player I should bet w.p. 1, and Player II should fold. In terms of P, this inequality becomes $P \leq (r+1)^{-n}$, so if $P \leq (r+1)^{-n}$, Player I should bluff initially w.p. p_1 and subsequently, if stage $j+1$ is reached, he should bluff w.p.

$$p_{j+1} = \frac{(1+r)^{n-j} - 1}{(r+1)^{n-j+1} - 1}. \tag{22}$$

The value of the game when the optimal bet sizes are used is

$$V_0 = a\left(\frac{P2^{n+1}s_n}{(s_n^{1/n} + s_0^{1/n})^n} - 1\right) \tag{23}$$

Example.

Suppose there are \$20 in the pot, so $s_0 = 10$, and suppose there are going to be three rounds of betting, so $n = 3$. If one player has \$260 in front of him and the other has \$330, then since $s_3 - s_0$ is the minimum of these two quantities, we have $s_3 = 260 + 10 = 270$. Then $s_3/s_0 = 27$ and $27^{1/3} = 3$, so from (18), $s_i = 10 \cdot 3^i$, so $s_1 = 30$ and $s_2 = 90$. Therefore, Player I should bet $s_1 - s_0 =$ \$20 on the first round. If II calls I should bet \$60$= s_2 - s_1$ on the second round. In this example, I bets the size of the pot at each round (including the third round), the same amount as in pot limit poker. To be in the all-strategies-active case, we require $P < 8/27$. If this is satisfied, Player II calls each bet of Player I w.p. $r = 1/2$; this agrees with (19). To find Player I's bluffing probabilities, we must specify P. If $P = 1/4$ for example, then $p_1 = (1/3)[(3/2)^3 - 1] = 19/24$. The subsequent betting probabilities (22) are

$$p_2 = \frac{(3/2)^2 - 1}{(3/2)^3 - 1} = 10/19 \qquad \text{and} \qquad p_3 = \frac{(3/2) - 1}{(3/2)^2 - 1} = 2/5.$$

Note that these probabilities are decreasing in the later rounds.

The proportion of the pot Player I bets on each round depends on the amount of the minimum table stakes. If this was \$70 instead of \$260, then the optimal bet would be half the size of the pot at each stage.

4. Basic Endgame With a Continuum Number of Rounds.

We see that Player I gets a definite advantage if he is allowed to split his betting over two rounds rather than betting the entire amount in one round. To see what sort of advantage I gets from a large number of rounds, we model the game as having a continuum number of rounds. We suppose that both players ante 1 unit each into the pot, and that the total amount to be bet is B, with an infinitesimal bet of dt being placed at time t for $0 < t < B$. Before play begins, Player I receives a winning card w.p. P and a losing card w.p. $1 - P$. If I has a winning card, he bets continuously throughout the whole interval $[0, B]$. If he has a losing card, he chooses a time $x \in [0, B]$ at which to stop betting. If $x = 0$, he passes initially, and if $x = B$ he bets throughout the whole interval. Player II chooses a time $y \in [0, B]$ at which to stop calling. For fixed choices of x and y, the expected payoff is

$$W(x,y) = \begin{cases} 1 + y & \text{if } 0 \le y < x \le B \\ P(1+y) - (1-P)(1+x) & \text{if } 0 \le x \le y \le B. \end{cases} \tag{1}$$

Thus if II stops calling before I stops betting, I wins the ante plus the total amount bet, namely $1 + y$. If I stops betting (with a losing card) before II stops calling, I wins $1 + y$ if he has the winning card and loses $1 + x$ if he has the losing card. We assume that x represents the last time Player I bets and y represents the last time player 2 calls, so that if $x = y$, the payoff is $P(1 + y) - (1 - P)(1 + x)$.

Let us analyze this game assuming the general principle that at all stages Player II will fold w.p. equal to the amount bet divided by the present pot size, which at time y is $dy/(2 + 2y + dy) \sim dy/(2 + 2y)$. We will see that this strategy makes I indifferent. If we let Y denote the random time at which Player II folds, then the general principle above implies that $\mathrm{P}\{Y = y | Y \ge y\} = dy/(2 + 2y)$. The quantity on the left is known as the failure rate of the distribution of Y. It is equal to $G'(y)/(1 - G(y))$, where $G(y)$ denotes the distribution function of Y. The above equation becomes $G'(y)/(1 - G(y)) = 1/(2(1 + y))$, which may also be written $-\frac{d}{dy}\log(1 - G(y)) = \frac{1}{2}\frac{d}{dy}\log(1 + y)$. Solving this differential equation for $G(y)$ and using the boundary condition $G(0) = 0$, we find

$$G(y) = 1 - \frac{1}{\sqrt{1+y}} \quad \text{for} \quad 0 < y < B. \tag{2}$$

Since $G(B) = 1$, we see that G gives probability $P_0 := 1/\sqrt{1+B}$ to the point B. This is the probability that Player II never folds. If we take the expectation of $W(x, Y)$ for fixed x we find

$$
\begin{aligned}
\int W(x,y)\,dG(y) &= \int_0^x (1+y)\,dG(y) + \int_x^B [P(1+y) - (1-P)(1+x)]\,dG(y) \\
&\qquad\qquad + \frac{1}{\sqrt{1+B}}[P(1+B) - (1-P)(1+x)] \\
&= (\sqrt{1+x} - 1) + P(\sqrt{1+B} - \sqrt{1+x}) \\
&\qquad\qquad - (1-P)(1+x)\left(\frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1+B}}\right) \\
&\qquad\qquad + \frac{1}{\sqrt{1+B}}[P(1+B) - (1-P)(1+x)] \\
&= 2P\sqrt{1+B} - 1 = 2\frac{P}{P_0} - 1
\end{aligned} \tag{3}
$$

independent of x. Since Player II can always keep her losses to be at most 1 by folding immediately, we suspect that the value of the game is the minimum of these, namely

$$
V = \begin{cases} 2(P/P_0) - 1 & \text{if} \quad P \le P_0 \\ 1 & \text{if} \quad P \ge P_0 \end{cases} \tag{4}
$$

To show that this is in fact the value, we consider a similar strategy for Player I. Let $F(x)$ denote the distribution function of random time at which Player I stops betting with a losing card, and take F to have a mass δ_0 at $x = 0$, zero probability on the interval $(z, B]$, for some $z \le B$, and a positive density on the interval $(0, z)$ of the following form:

$$
F(x) = \delta_0 + c(1 - \frac{1}{\sqrt{1+x}}) \quad \text{for} \quad 0 \le x \le z, \tag{5}
$$

where the parameters $\delta_0 \ge 0$, $c > 0$ and $z > 0$ are restricted so that $F(z) = 1$. For fixed y with $0 \le y \le z$, the expectation of $W(X, y)$ is

$$
\int W(x,y)\,dF(x) = (1+y)(1 - (1-P)(\delta_0 + c)) + (1-P)(c - \delta_0) \tag{6}
$$

and for $z \le y \le B$, we have

$$
\int W(x,y)\,dF(x) = P(1+y) - (1-P)(\delta_0 + c(\sqrt{1+z} - 1)). \tag{7}
$$

Since (7) is increasing in y, Player II will never choose a y larger than z. Expression (6) is a constant in $y \in [0, z]$ provided $(1-P)(\delta_0 + c) = 1$ so Player I can keep the value of the game to at least $(1-P)(c - \delta_0)$. So I chooses $\delta_0 = (1/(1-P)) - c$ and proceeds to choose c and z to make c as large as possible, subject to the condition that F, which is now $F(x) =$

$(1/(1-P)) - (c/\sqrt{1+x})$ for $0 \le x \le z$, is still a distribution function. If $P\sqrt{1+B} \le 1$, we may choose $z = B$ and $c = (P/(1-P))\sqrt{1+B}$, since $\delta_0 = (1 - P\sqrt{1+B})/(1-P)$ is still nonnegative. In this case the value is $(1-P)(c-\delta_0) = 2P\sqrt{1+B} - 1$, the same as found in (4). But if $P\sqrt{1+B} \ge 1$, we must choose $\delta_0 = 0$ and $c = 1/(1-P)$, when $z = (1/P^2) - 1$. In this case the value is $(1-P)(c-\delta_0) = 1$, also found in (4).

Summary of the Solution. Let $P_0 = 1/\sqrt{B+1}$.

If $P \le P_0$, then the value is $2(P/P_0) - 1$. Player I has an optimal strategy,

$$F(x) = \frac{1}{1-P} - \frac{P}{1-P}\frac{\sqrt{1+B}}{\sqrt{1+x}} \qquad \text{for } 0 \le x \le B,$$

with mass $(1 - P\sqrt{1+B})/(1-P)$ at $x = 0$. Player II has an optimal strategy

$$G(y) = 1 - \frac{1}{\sqrt{1+y}} \qquad \text{for } 0 \le y \le B,$$

with mass P_0 at $y = B$.

If $P \ge P_0$, then the value is 1. Player I has an optimal strategy,

$$F(x) = \frac{1}{1-P}\left(1 - \frac{1}{\sqrt{1+x}}\right) \qquad \text{for } 0 \le x \le \frac{1}{P^2} - 1.$$

Player II has an optimal strategy that gives mass 1 to the point $y = 0$ (i.e. fold immediately).

It may be noted that when $P\sqrt{1+B} \le 1$, the hazard rate of Player I's optimal strategy (i.e. $F'(x)/(1-F(x))$) is independent of P. Of special interest is the fact that if Player I is bluffing, he will stop betting before reaching B; that is his strategy gives zero probability to the point B.

5. Endgame with Information for Player II.

We modify Basic Endgame by allowing Player II to receive information on Player I's card. This provides a model of situations that occur regularly in poker. For example, suppose Player I needs a four or a nine to complete a straight, or a spade to complete a flush. Player II's hand may contain a four or a spade unknown to Player I, giving Player II some hidden information on the chances that Player I has a straight or a flush. This situation is modeled as follows.

The game is two-person, zero-sum. Player I receives a winning card, W, w.p. P, and a losing card, L, w.p. $1-P$, where $0 < P < 1$. Then Player I observes his card and must either check or bet a fixed amount $b > 0$. If Player I checks, the game ends and Player I wins or loses the ante $a > 0$, depending on whether he has W or L.

Suppose Player I bets. Then Player II is allowed to observe a random variable Y whose distribution depends on the card received by Player I. We may assume without loss of generality that the density of Y with respect to a σ-finite measure, μ, exists. Let $f_W(y)$ denote the density of Y if I has W, and $f_L(y)$ denote the density of Y if I has L. Based on the observation Y, II must either call or fold. If Player II folds, the game ends and I wins a from II. If Player II calls, the game ends and Player I wins or loses $a+b$ depending on whether I has W or L. It is assumed that both players know a, b, P and f_W, f_L and μ.

It is clear that Player I may as well bet whenever he holds W. We assume without loss of generality that the rules of the game require this. Then a pure strategy for I is just a rule telling him what to do when he receives an L. Therefore, I has just two pure strategies. He may bet with a losing card, the *bluff* strategy, or he may check with a losing card, the *honest* strategy.

A pure strategy for II is a rule telling her whether to fold or call for each possible value of Y that may be observed. A behavioral strategy for Player II is a function $\phi(y)$ satisfying $0 \le \phi(y) \le 1$ for all y, with the understanding that if II observes $Y = y$ she folds w.p. $\phi(y)$ and calls with probability $1-\phi(y)$. In statistical parlance, ϕ is called a test.

Let $E_W\phi(Y) = \int \phi(y) f_W(y)\, d\mu(y)$. Then $E_W\phi(Y)$ represents the probability that II folds given I has a high card and bets. Similarly, $E_L\phi(Y) = \int \phi(y) f_L(y)\, d\mu(y)$ represents the probability that II folds given I has a low card and bets. The expected payoff to Player I if he uses one of his pure strategies and II uses ϕ is

$$\begin{aligned} V(\mathit{bluff}, \phi) &= P[(a+b)\mathrm{E}_W(1-\phi(Y)) + a\mathrm{E}_W\phi(Y)] \\ &\qquad + (1-P)[-(a+b)\mathrm{E}_L(1-\phi(Y)) + a\mathrm{E}_L\phi(Y)] \\ &= (2P-1)(a+b) - Pb\mathrm{E}_W\phi(Y) + (1-P)(2a+b)\mathrm{E}_L\phi(Y) \\ V(\mathit{honest}, \phi) &= P[(a+b)\mathrm{E}_W(1-\phi(Y)) + a\mathrm{E}_W\phi(Y)] - (1-P)a \\ &= (2P-1)a + Pb - Pb\mathrm{E}_W\phi(Y). \end{aligned} \tag{1}$$

If $\mathrm{E}_L\phi(Y)$ is fixed equal to some number α, then II will choose ϕ subject to this constraint to maximize $\mathrm{E}_W\phi(Y)$ since both $V(\mathit{bluff}, \phi)$ and $V(\mathit{honest}, \phi)$ are decreasing in $\mathrm{E}_W\phi(Y)$. Such a ϕ is called a best test of size α for testing the hypothesis $H_0 : f_L(y)$ against $H_1 : f_W(y)$. Player II may restrict her attention to such ϕ. The Neyman-Pearson Lemma states that for fixed $0 < \alpha \le 1$, the test of the form

$$\phi_\alpha(y) = \begin{cases} 0 & \text{if } f_W(y) < k f_L(y) \\ \gamma & \text{if } f_W(y) = k f_L(y) \\ 1 & \text{if } f_W(y) > k f_L(y) \end{cases} \tag{2}$$

is a best test of size α when $k \ge 0$ and γ are chosen so that $\mathrm{E}_L\phi(Y) = \alpha$. In addition, corresponding to $k = \infty$, the test

$$\phi_0(y) = \begin{cases} 1 & \text{if } f_L(y) = 0 \\ 0 & \text{if } f_L(y) > 0 \end{cases} \tag{3}$$

is a best test of size $\alpha = 0$. Player II may restrict attention to the tests ϕ_α for $0 \le \alpha \le 1$.

Player II's optimal strategy is to choose α so that the maximum of $V(\mathit{bluff}, \phi_\alpha)$ and $V(\mathit{honest}, \phi_\alpha)$ is a minimum. The difference in the payoffs is $V(\mathit{bluff}, \phi_\alpha) - V(\mathit{honest}, \phi_\alpha) = (1-P)((2a+b)\alpha - b)$. This is linear increasing in α with a unique root,

$$\alpha_0 = b/(2a+b) \tag{4}$$

in $(0,1)$. Moreover, $\mathrm{E}_W \phi_\alpha(Y)$ is nondecreasing in α (in fact concave, and increasing as long as it is less than 1), so $V(\mathit{honest}, \phi_\alpha)$ is nonincreasing in α. Hence, there are only two cases, depending on the sign of s_r, the right slope of $V(\mathit{bluff}, \phi_\alpha)$ at $\alpha = \alpha_0$.

Case 1: $s_r > 0$. In this case, ϕ_{α_0} is optimal for Player II. It is interesting to note that this strategy does not depend on P. If s_l denotes the left slope of $V(\mathit{honest}, \phi_\alpha)$, then $s_l \le 0$ and an optimal strategy for Player I is to mix *bluff* and *honest* in the proportions $|s_l|$ and s_r.

Case 2: $s_r \le 0$. In this case, it is optimal for Player I to bluff all the time. If α_1 denotes the value of α that minimizes $V(\mathit{bluff}, \phi_\alpha)$, then $\alpha_1 > \alpha_0$ and ϕ_{α_1} is an optimal strategy for Player II. In this case, the optimal strategy for II may be taken to be nonrandomized.

As noted in Ferguson (1967), a sufficient condition for s_r to be positive is that $P < 1/2$. This may be seen as follows. Let $g(\alpha) = \mathrm{E}_W \phi_\alpha(Y)$. Using $\mathrm{E}_L \phi_\alpha(Y) = \alpha$ in (1), we have $V(\mathit{bluff}, \phi_\alpha) = (2P-1)(a+b) - Pbg(\alpha) + (1-P)(2a+b)\alpha$. Letting $g'(\alpha)$ denote the right slope of $g(\alpha)$, we have $s_r = -Pbg'(\alpha_0) + (1-P)(2a+b) = P(2a+b)[(1-P)/P - \alpha_0 g'(\alpha_0)$. But since g is concave and $g(0) \ge 0$, we have $g'(\alpha_0) \le g(\alpha_0)/\alpha_0$. Therefore, if $P < 1/2$, we have $s_r > 0$.

It is interesting to compare this solution with that of Basic Endgame. In the latter, the cutoff between Case 1 and Case 2 is $P = (2a+b)/(2a+2b)$, and while the cutoff between Case 1 and Case 2 in the above solution may be greater or less than this, it is always at least $1/2$. In Case 1 of Basic Endgame, Player II always folds with probability $\alpha_0 = b/(2a+b)$, while in the above solution, II folds w.p. α_0 when Player I has L (i.e. $\mathrm{E}_L \phi_{\alpha_0}(Y) = \alpha_0$), and w.p. $\mathrm{E}_W \phi_{\alpha_0}(Y) > \alpha_0$ when Player I has W. In other words, Player I can detect that Player II is using his information only by noting that II is calling I's winning hands less often.

Although Player I's strategy depends on P, one feature of this strategy is independent of P, namely the probability that he has L given that he has bet in Case 1. Using $s_r = -Pbg'(\alpha_0) + (1-P)(2a+b)$ and $s_l = -Pbg'(\alpha_0)$, we find the probability that I bets with a L is $|s_l|/(s_r + |s_l|) = Pbg'(\alpha_0)/((1-P)(2a+b)) = P\alpha_0 g'(\alpha_0)/(1-P)$. Thus we have $P(\text{I has } L | \text{I bets}) = \alpha_0 g'(\alpha_0)/(1 + \alpha_0 g'(\alpha_0))$, independent of P.

The Binary Case.

As a simple illustration of the computations involved, we consider the case where the observation Y takes only two values, say 0 and 1, and $f_L(1) = p_L$, $f_L(0) = 1 - p_L$, $f_W(1) = p_W$ and $f_W(0) = 1 - p_W$. If $p_L = p_W$, then the observation of Y gives no information about the card Player I has, and the problem reduces to Basic Endgame. Otherwise, we may assume without loss of generality that $p_W > p_L$. Then Player II prefers to see $Y = 0$ since then it is less likely Player I has W. We refer to $Y = 0$ as good hands of Player II, and $Y = 1$ as poor hands.

First we find the best test of size α. For $0 \le \alpha < p_L$, we must set $k = p_W/p_L$ and we have

$$\phi_\alpha(y) = \begin{cases} 0 & \text{if } y = 0 \\ \gamma & \text{if } y = 1 \end{cases} \tag{5}$$

where for $\mathrm{E}_L\phi_\alpha(Y) = \gamma p_L$ to be equal to α, we require $\gamma = \alpha/p_L$. This is the strategy: *Call with a good hand. With a poor hand, fold w.p.* $\gamma = \alpha/p_L$ *and call w.p.* $1 - \gamma$.

For $p_L < \alpha \le 1$, we must set $k = (1 - p_W)/(1 - p_L)$ and we have

$$\phi_\alpha(y) = \begin{cases} \gamma & \text{if } y = 0 \\ 1 & \text{if } y = 1 \end{cases} \tag{6}$$

where for $\mathrm{E}_L\phi_\alpha(Y) = p_L + (1 - p_L)\gamma$ to be equal to α, we require $\gamma = (\alpha - p_L)/(1 - p_L)$. This is the strategy: *Fold with a poor hand. With a good hand, fold with probability* $\gamma = (\alpha - p_L)/(1 - p_L)$ *and call w.p.* $1 - \gamma$.

For $\alpha = p_L$, we may put $k = 1$ and find

$$\phi_\alpha(y) = \begin{cases} 0 & \text{if } y = 0 \\ 1 & \text{if } y = 1. \end{cases} \tag{7}$$

This is the strategy: *Call with a good hand. Fold with a poor hand.*

To implement the above solution, we need s_r, the right slope of $V(\mathit{bluff}, \phi_\alpha)$, and s_l, the left slope of $V(\mathit{honest}, \phi_\alpha)$. We first find

$$\mathrm{E}_W\phi_\alpha(Y) = \begin{cases} \alpha p_W/p_L & \text{for } 0 \le \alpha \le p_L \\ p_W + (\alpha - p_L)(1 - p_W)/(1 - p_L) & \text{for } p_L < \alpha \le 1. \end{cases} \tag{8}$$

From this we may find

$$s_r(\alpha) = \begin{cases} -Pbp_W/p_L + (1 - P)(2a + b) & \text{for } 0 \le \alpha < p_L \\ -Pb(1 - p_W)/(1 - p_L) + (1 - P)(2a + b) & \text{for } p_L \le \alpha < 1. \end{cases} \tag{9}$$

and

$$s_l(\alpha) = \begin{cases} -bPp_W/p_L & \text{for } 0 < \alpha \le p_L \\ -bP(1 - p_W)/(1 - p_L) & \text{for } p_L < \alpha \le 1. \end{cases} \tag{10}$$

We may state the solution as follows. Let

$$\alpha_0 = \frac{b}{2a + b} \qquad \beta_0 = \frac{p_L}{p_W}\frac{1 - P}{P} \qquad \beta_1 = \frac{1 - p_L}{1 - p_W}\frac{1 - P}{P}$$

and note that $\beta_0 < \beta_1$ since we are assuming $p_L < p_W$.

1. If $\alpha_0 \geq \beta_1$, then
   it is optimal for I to bluff, and
   it is optimal for II to fold.
   The value is a.

2. If $\alpha_0 < \beta_1$ and $\alpha_0 \geq p_L$, then
   it is optimal for I to bluff w.p. α_0/β_1, and
   it is optimal for II to fold with a poor hand and
   to fold w.p. $(\alpha_0 - p_L)/(1 - p_L)$ with a good hand.
   The value is $a[1 - 2(1 - P)(1 - (\alpha_0/\beta_1))]$.

3. If $\beta_0 \leq \alpha_0 < \beta_1$ and $\alpha_0 < p_L$, then
   it is optimal for I to bluff, and
   it is optimal for II to fold with a poor hand and
   to call with a good hand.
   The value is $(2P - 1)(a + b) - p_L(1 - P)b((\alpha_0/\beta_0) - 1)(2a + b)$.

4. If $\alpha_0 < \beta_0$ and $\alpha_0 < p_L$, then
   it is optimal for I to bluff w.p. α_0/β_0, and
   it is optimal for II to call with a good hand and
   to fold w.p. α_0/p_L with a poor hand.
   The value is $(2P - 1)(a + b) + b(1 - P)(1 - (\alpha_0/\beta_0))$.

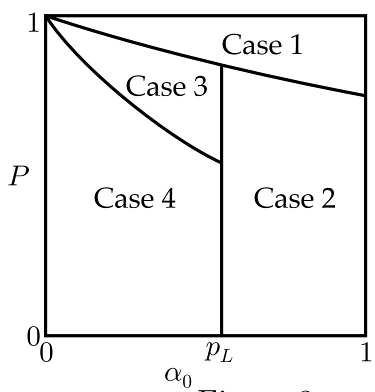

Figure 3.

Figure 3 shows the regions of (α_0, P) for the four cases when $p_L = .35$ and $p_W = .85$.

The mixed strategy cases (Case 2 and Case 4) may be derived from the solution to Basic Endgame as follows. Player I will be indifferent between betting and folding with an L if Player II folds with overall probability α_0. So II should choose her strategy in such a way that her overall probability of folding is α_0. In Case 4 if I has an L, II folds w.p. $p_L \cdot (\alpha_0/p_L)$ which is α_0. Also in Case 2 if I has an L, II folds w.p. $p_L + (1 - p_L)(\alpha_0 - p_L)/(1 - p_L)$ which is also α_0.

Similarly for Player I. In Case 4, he should play to keep II indifferent if she has a poor hand, while in Case 2 he should play to keep her indifferent if she has a good hand. If Player II has a poor hand, she evaluates the probability that I has a W as

$P_1 = Pp_W/(Pp_W + (1-P)p_L) = 1/(1+\beta_0)$, so I should replace P in his strategy for Basic Endgame with P_1. This is in fact what he does since $\alpha_0 P_1/(1-P_1) = \alpha_0/\beta_0$. If Player II has a good hand, she evaluates the probability that I has a W as $P_1 = P(1-p_W)+(1-P)(1-p_L)$, so I should bluff w.p. $\alpha_0 P_0/(1-P_0) = \alpha_0/\beta_1$, which is what he does.

Examples.

Example 1. Consider the introductory example of the last round of stud poker with Player I having a 5, 6, 7 and 8, not suited, showing and Player II having 2, 3, K and K showing. For the purposes of this example we shall take the probabilities, P, p_L, and p_W, to be those assuming that the down card was dealt last. Since there are exactly 8 cards out of the remaining 44 cards that can give Player I a straight, we have $P = 8/44 = 2/11$. Similarly, $p_L = 35/43$ and $p_W = 36/43$. Then we may compute $\beta_0 = (35/36)\cdot(9/2) = 35/8$, and $\beta_1 = (8/7)\cdot(9/2) = 36/7$. Suppose we are playing pot limit poker so that $b = 2a$ and $\alpha_0 = 1/2$. Then $\alpha_0 < \beta_0$ and $\alpha_0 < p_L$ so we are in Case 4. If he doesn't have the straight, Player I should bluff w.p. $\alpha_0/\beta_0 = 4/35$. With a good card (4 or 9), Player II should call. With a poor card, she should fold w.p. $\alpha_0/p_L = 43/70$.

Example 2. An interesting feature of this game is that, unlike Basic Endgame, it may be optimal to bet less than the maximum allowable bet. One can guess that this will occur in situations where Player II can sometimes, though rarely, get sure information that Player I has a losing card. I can gain by bluffing reasonably often, but with a high enough bet size this cannot be optimal because Player II will call only when she knows she will win. Here is an example.

Suppose $P = 1/2$, $p_W = 1$ and $p_L = 1-\epsilon$ for some small $\epsilon > 0$. This means that Player II will occasionally (probability ϵ) get accurate information that Player I has a losing card. Then $\beta_1 = \infty$ so case 1 does not occur. Moreover $\beta_0 = p_L$ so Case 3 does not occur. If $\alpha_0 \geq p_L$ (Case 2), then I does not bluff and the value is 0. If $\alpha_0 < p_L$ (Case 4), then I bluffs w.p. α_0/p_L, and II calls with a good hand and folds w.p. α_0/p_L with a poor hand. The value is $(b/2)(1-(\alpha_0/p_L)) = b(p_L - (b/(2a+b))/(2p_L)$. This is positive for α_0 in the interval $[0, p_L]$, and has a maximum at $b = 2a((1/\sqrt{\epsilon})-1)$. For example, if $\epsilon = 1/4$, then the optimal bet size is $b = 2a$, the bet size for pot limit poker.

Example 3. The phenomenon of finite optimal bet size is frequent in this model. We take another example with continuous observations for Player II. Suppose the random variable Y has a uniform distribution when Player I has a losing card, i.e. $f_L(y) = 1$ on the set $(0,1)$, and suppose Y has a triangular distribution with $f_W(y) = 2(1-y)$ on $(0,1)$ when I has a winning card. The best tests of size α are simply

$$\phi_\alpha(y) = \begin{cases} 1 & \text{if } y \leq \alpha \\ 0 & \text{if } y > \alpha. \end{cases}$$

Then $g(\alpha) = \mathrm{E}\phi_\alpha(Y) = (2-\alpha)\alpha$. We are in the case of a positive slope of $V(\mathit{bluff}, \phi_\alpha)$ at $\alpha = \alpha_0 = b/(2a+b)$, so ϕ_{α_0} is optimal for Player II. The value may be computed as $V(\mathit{honest}, \phi_{\alpha_0}) = (2P-1)a + 4a^2Pb/(2a+b)^2$. As a function of b, the value has a maximum at $b = 2a$, again pot limit poker.

6. Endgame with Imprecise Information for Player I.

We consider the generalization of basic endgame achieved by allowing the card received by Player I to indicate only his probability of winning. This models situations in which there are still cards to be drawn, which with some probability may change a winning hand into a losing one. We model this by denoting the card Player I receives by the probability of win. Thus, if he receives a card marked p, then p is his probability of win. The distribution of p on $[0, 1]$ is arbitrary however, and may be discrete or continuous. The distribution function of p is denoted by F. In Basic Endgame, p is either 0 or 1, and the probability that $p = 1$ is P. It is assumed that both players know F but only Player I learns the value of p. Other than this, the form of the game with one possible bet and then a call or fold is the same as it was in Basic Endgame.

Player II has the same two pure strategies of fold or call. But now Player I may use a distinct strategy for each card he receives. A mixed strategy for Player I is a function, $\phi(p)$, that denotes the probability that Player I bets when he observes that his probability of win is p. We have $0 \le \phi(p) \le 1$ for all p. The expected payoff to Player I if he uses ϕ and II uses one of her pure strategies is

$$\begin{aligned}
V(\phi, \mathit{fold}) &= \mathrm{E}[\phi(p)a - (1-\phi(p))a(2p-1)] = 2a\mathrm{E}(1-p)\phi(p) + a(2\mu - 1) \\
V(\phi, \mathit{call}) &= \mathrm{E}[\phi(p)(2p-1)(a+b) + (1-\phi(p))a(2p-1)] \\
&= b\mathrm{E}\phi(p) - 2b\mathrm{E}(1-p)\phi(p) + a(2\mu - 1)
\end{aligned}$$

where $\mu = \mathrm{E}p$ denotes the overall probability that I wins. Out of the class of ϕ (with $0 \le \phi(p) \le 1$) such that $\mathrm{E}(1-p)\phi(p)$ is held fixed, I will choose ϕ to maximize $\mathrm{E}\phi(p)$, since that will maximize $V(\phi, \mathit{call})$ with $V(\phi, \mathit{fold})$ held fixed. This is equivalent to maximizing $\mathrm{E}p\phi(p)$ out of the class of ϕ such that $\mathrm{E}\phi(p)$ is held fixed equal to some number α. By the Neyman-Pearson Lemma, we may find such a ϕ of the form

$$\phi_\alpha(p) = \begin{cases} 1 & \text{if } p > k \\ \gamma & \text{if } p = k \\ 0 & \text{if } p < k \end{cases}$$

where k and γ are chosen so that $\mathrm{E}\phi(p) = \alpha$. Player I may restrict his choice of strategy to the class of ϕ_α. This shows that Player I will bet with cards having a high value of p and check with cards having a low value of p. Any randomization will occur only for those p on the boundary between betting and checking. If the distribution function F is continuous, Player I will have an optimal pure strategy. Note that both $V(\phi, \mathit{fold})$ and $V(\phi, \mathit{call})$ are increased by putting $\phi(p) = 1$ for $p \ge 1/2$. Therefore in the model, Player I should always bet if his probability of win is at least $1/2$.

Let $g(\alpha) = \mathrm{E}p\phi_\alpha(p)$. Then $g(\alpha)$ is continuous, concave and increasing on $[0, 1]$. It is apparent that $V(\phi_\alpha, \mathit{fold})$ is a nondecreasing convex function of α in [0,1], while $V(\phi_\alpha, \mathit{call})$ is concave and increasing in α as long as $k > 1/2$ and nonincreasing thereafter. Moreover, these functions are equal at $\alpha = 0$ and there is at most one other value of α in $(0, 1]$ at which they are equal.

From these observations, we may write down the solution of the game. The form of the optimal strategies differs in three different regions of the space of parameters.

Case 1. $\mu \geq (2a+b)/(2a+2b)$. In this case, $V(\phi_\alpha, \mathit{fold}) \leq V(\phi_\alpha, \mathit{call})$ for all values of α. Therefore, folding is optimal for II, betting always is optimal for I and the value is a.

Let $\alpha_0 = \mathrm{P}(p \geq 1/2)$ and if $\alpha_0 > 0$, let $\mu_0 = \mathrm{E}(p|p > 1/2) = \mathrm{E}p\phi_{\alpha_0}(p)/\alpha_0$. Note that $V(\phi_\alpha, \mathit{call})$ takes on its maximum value at $\alpha = \alpha_0$ and that ϕ_{α_0} is the indicator of the set $\{p \geq 1/2\}$. Also note that $\mu_0 \geq \mu$.

Case 2. $\alpha_0 = 0$, or $\alpha_0 > 0$ and $\mu_0 \leq (2a+b)/(2a+2b)$. In this case, $V(\phi_{\alpha_0}, \mathit{call}) \leq V(\phi_{\alpha_0}, \mathit{fold})$, so that the maximum of the minimum of these two functions occurs at α_0. Therefore, calling is optimal for II, and ϕ_{α_0} is optimal for I. If $\alpha_0 = 0$, then $V(\phi_\alpha, \mathit{call})$ is decreasing in α, so that checking always is optimal for I. Otherwise, the strategy that bets if and only if $p \geq 1/2$ is optimal for I. The value is $\alpha_0(2b\mu_0 - 1) + a(2\mu - 1)$.

The last case is the main case, requiring mixed strategies for Player II. The minimax value of α occurs at the point of intersection of $V(\phi_\alpha, \mathit{fold})$ and $V(\phi_\alpha, \mathit{call})$, call it α_1. Then α_1 satisfies the equation $g(\alpha_1)/\alpha_1 = (2a+b)/(2a+2b)$.

Case 3. $\mu < (2a+b)/(2a+2b) < \mu_0$. In this case, the maximin occurs at α_1. Therefore, mixing fold and call in the proportions $b - 2bg'(\alpha_1) : 2a - 2ag'(\alpha_1)$ is optimal for II, and ϕ_{α_1} is optimal for I. (Here $g'(\alpha)$ represents any value between the left derivative and the right derivative at α.) The value is $2a(\alpha_1 - g(\alpha_1)) + a(2\mu - 1)$.

The Binary Case.

As an illustration, consider the case where F gives mass to only two points. Suppose F gives mass π to p_H and mass $1-\pi$ to p_L, where $p_L < p_H$. Thus, Player I receives one of two cards, a high card giving him probability p_H of winning, and a low card giving him probability p_L of winning. The probability he receives a high card is $0 < \pi < 1$. The probability that I wins with the card he receives is $\mu = \pi p_H + (1-\pi)p_L$. Basic Endgame occurs when $p_L = 0$, $p_H = 1$, and $\mu = \pi = P$. We let $Q_0 = (2a+b)/(2a+2b)$.

If both p_L and p_H are in $[0, 1/2]$, then it is optimal for I to check with either card and it is optimal for II to call (Case 2 with $\alpha_0 = 0$). The value is $a(2\mu - 1)$.

If both p_L and p_H are in $[1/2, 1]$, then $\alpha_0 = \mathrm{P}(p \geq 1/2) = 1$ and it is optimal for I to bet with either card. It is optimal for II to fold if $\mu \leq Q_0$ (Case 1, value $= a$) and to call if $\mu \geq Q_0$ (Case 2, value $= (a+b)(2\mu - 1)$).

Suppose now that $p_L < 1/2 < p_H$. Then $\alpha_0 = \pi$ and $\mu_0 = \mathrm{E}(p|p > 1/2) = p_H$.

If $Q_0 \leq \mu$, then it is optimal for II to fold and for I to bet with either card (Case 1, value $= a$). If $p_H \leq Q_0$, then it is optimal for II to call and it is optimal for I to bet with a high card and check with a low card (Case 2 with $0 < \alpha_0 < 1$ and value $= a(2\mu - 1) + b\pi(2p_H - 1)$).

There remains to consider $\mu < Q_0 < p_H$ (Case 3). This requires computation of ϕ_α

and the payoff functions. The strategies ϕ_α are easily found.

$$\text{For } \alpha \le \pi, \quad \phi_\alpha(p) = \begin{cases} \alpha/\pi & \text{if } p = p_H \\ 0 & \text{if } p = p_L \end{cases}$$
$$\text{For } \alpha \ge \pi, \quad \phi_\alpha(p) = \begin{cases} 1 & \text{if } p = p_H \\ (\alpha-\pi)/(1-\pi) & \text{if } p = p_L \end{cases}$$

From this we may find $g(\alpha) = \mathrm{E}p\phi_\alpha(p)$.

$$g(\alpha) = \begin{cases} \alpha p_H & \text{for } \alpha \le \pi \\ \pi p_H + (\alpha-\pi)p_L & \text{for } \alpha \ge \pi \end{cases}$$

and from this we may find the expected payoffs.

$$V(\phi_\alpha, \mathit{fold}) = 2a(\alpha - g(\alpha)) + a(2\mu - 1)$$
$$V(\phi_\alpha, \mathit{call}) = b(2g(\alpha) - \alpha) + a(2\mu - 1).$$

The value, α_1, at which these intersect is that value of α in $[\pi, 1]$ such that $2a(\alpha - g(\alpha)) = b(2g(\alpha) - \alpha)$. This reduces to
$$\alpha_1 = \frac{p_H - p_L}{Q_0 - p_L}\pi.$$
I's optimal strategy is ϕ_{α_1}. This strategy calls for betting with a high card and randomizing with a low card, betting w.p. $(\alpha_1 - \pi)/(1-\pi)$ and checking otherwise. The optimal strategy for Player II is to mix calling and folding in proportions $|s_c| : s_f$, where s_c and s_f are the slopes at α_1 of the payoff functions for calling and folding. Since $s_1 = 2a(1 - p_L)$ and $s_2 = b(2p_L - 1)$, we have that II should call with probability

$$\frac{b(1-2p_l)}{(2a+b) - (2a+2b)p_l} = \frac{1-Q_0}{Q_0}\frac{1-2p_L}{1-p_L}$$

and fold otherwise. The value is

$$2a(\alpha_1 - g(\alpha_1)) + a(2\mu - 1) = 2a\pi\frac{(p_H - p_L)(1-Q_0)}{Q_0 - p_L} + a(2\mu - 1).$$

It is interesting to note that II's optimal strategy in Case 3 (the general case) does not depend on π or p_H. Player II's strategy depends only on Q_0 and p_L. It should also be noted that in all cases Player I's optimal strategy does not randomize with a high card.

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