Obliged Sums of Games

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1. Introduction. Let g be an impartial combinatorial game. In such a game, there are two players, I and II, there is an initial position, and players alternate moving with Player I starting. The possible legal moves from one position to another is determined by the rules of the game and is independent of the player whose turn it is to move. The game ends when there are no further legal moves. It is assumed that the game satisfies the ending condition, that the game ends in a finite number of moves no matter how it is played. Under the normal play rule, the last player to move wins. Under the misère play rule, the last player to move loses. As a reference, see Winning Ways, [1].

Now, let $g_1, g_2, \ldots, g_n$ be $n \geq 1$ such games, each with its own rules. In the disjunctive sum of these games, $\boldsymbol{G} = g_1 + g_2 + \cdots + g_n$, the player to move chooses one of the games and makes a move in it. Then the other player chooses one of the games, perhaps a different game or possibly the same game, and makes a move in it. The sum of the games continues until there are no moves left in any of the component games, and the player who moved last is the winner under the normal play rule and the loser under the misère play rule. There is a vast and deep theory to help solve disjunctive sums of games under the normal play rule. Under the misère play rule, however, the problems are much more complex and solutions are achievable only in special circumstances.

Here we consider a different sum of these games, suggested by N.A. Ashrafi Payman and Keivan Borna, [2]. We denote the sum by $\boldsymbol{G} = g_1 \oplus g_2 \oplus \cdots \oplus g_n$, and call it the obliged sum. In the obliged sum, if a player makes a move in a component game, the players are then obliged to continue that component game until there are no further moves remaining in it. Then the player whose turn it is to move, chooses another of the component games and makes a move in it, with the same obligations until all games are finished. Again the winner is determined by the normal play rule or the misère play rule, whichever rule is in force. In contrast to the disjunctive sum of games, the solution to an obliged sum of games is equally difficult under the normal play rule and the misère play rule. In fact, the two problems must be solved together.

2. The Structure Theorem for Obliged Sums of Games. For solving obliged sums of games, we need to distinguish between the outcomes of P positions for normal play and P positions for misère play. Let us use $Pnor$ to denote a P position for normal play, and $Pmis$ to denote a P position for misère play. Similarly, let $Nnor$ and $Nmis$ denote the N positions for normal and misère play respectively. (A P position is one from which you can win if it is the opponent's turn to move. An N position is one from which you can win if it is your turn to move.)

For a given obliged sum game, G, we define the outcome or value of $\boldsymbol{G}$, denoted by $Val(\boldsymbol{G})$, to be the pair of outcomes for normal and misère play. Thus for each such $\boldsymbol{G}$, there are four possible values, namely, $(Pnor, Pmis)$, $(Pnor, Nmis)$, $(Nnor, Pmis)$ and $(Nnor, Nmis)$. For simplicity, we replace $Pnor$ and $Pmis$ by 0 and $Nnor$ and $Nmis$ by

1 and drop the parentheses. Thus, $Val(\boldsymbol{G}) = 01$ indicates that $\boldsymbol{G}$ is a P position under normal play, and an N position under misère play.

In the following theorem, we see that the value of an obliged game is determined by the values of its component games. For a given obliged game, $\boldsymbol{G} = g_1 \oplus g_2 \oplus \cdots \oplus g_n$, we define m_{ij} for $i = 0$ or 1, and $j = 0$ or 1, to be the number of component games, g_k, for which $Val(g_k) = ij$.

Theorem 1. *Let $\boldsymbol{G} = g_1 \oplus g_2 \oplus \cdots \oplus g_n$ be an obliged sum of games. Let m_{ij} denote the number of component games with value ij. Then,*
(a) If $m_{11} > 0$, then $Val(\boldsymbol{G}) = 11$.
(b) If $m_{11} = 0$ and $m_{00} > 0$, then $Val(\boldsymbol{G}) = 11$ if m_{10} is odd, and $Val(\boldsymbol{G}) = 00$ if m_{10} is even.
(c) If $m_{11} = 0$ and $m_{00} = 0$, then $Val(\boldsymbol{G}) = 10$ if m_{10} is odd, and $Val(\boldsymbol{G}) = 01$ if m_{10} is even.

Proof. Note that the theorem makes no mention of the number m_{01}, and indeed parts (a) (b) and (c) are valid independent of the value of m_{01}. The reason for this is as follows. Suppose the player to move moves in a component game, g_k, with $Val(g_k) = 01$. That player can win the misère version of the game g_k, thus keeping the move. But also his opponent can win the normal version of the game g_k, forcing the original player to keep the move. No matter which player initiates play in g_k, one of these two strategies must be optimal, so the game g_k will result, under optimal play, with the player to move keeping the move. Thus, whenever such a game is played optimally, it is completed with the same player to move. We may as well assume that there are no component games g_k with value 01.

To prove part (c), note that if all components have value 10, then an argument similar to the above shows that under optimal play, the player whose turn it is to move will change after each component game is finished. If m_{10} is even, the player with the move at the end will be the original player with the move; hence the value of the obliged sum, $\boldsymbol{G}$, is 01, since the original player will have lost the normal version and won the misère version. Similarly if m_{10} is odd, the value of $\boldsymbol{G}$ is 10.

Now note that if the player to move moves in a component game, g_k, with value $Val(g_k) = 00$, his opponent can move to win either the normal or misère version of g_k, whichever will win him the obliged sum, $\boldsymbol{G}$. Thus, neither player will start play in a component game with value 00 unless forced to so.

To prove part (b), suppose the only components left have value either 00 or 10, with $m_{00} > 0$. This time the players may as well play the components with value 10, and then the player whose turn it is to move loses both normal and misère versions of $\boldsymbol{G}$. But the player who wins the normal version of the last g_k with value 10. With an odd number of g_k's of value 10, this is the player who starts first, in which case the value of $\boldsymbol{G}$ is 11. If $m_{0,0}$ is even, then the value is 00.

To prove part (a), suppose one of the component games, say g_k, has value 11. Let $\boldsymbol{G}^{(k)}$ denote the obliged game $\boldsymbol{G}$ with component game g_k removed. Since the player to move can win either the normal or the misère version of g_k, he can choose whether to be first or second to move in the game $\boldsymbol{G}^{(k)}$. Then depending on the value of $\boldsymbol{G}^{(k)}$, he can choose to win either the normal or misère version of $\mathbf{G}$ by moving appropriately in g_k. Thus $Val(\boldsymbol{G}) = 11$. This gives part (a) and completes the proof.

In terms of the *Pnor* and *Pmis* positions of $\boldsymbol{G}$, this theorem has a simpler statement.

Corollary. *Let $\boldsymbol{G} = g_1 \oplus g_2 \oplus \cdots \oplus g_n$ be an obliged sum of games. Let m_{ij} denote the number of component games with value ij. Then,*
(a) $\boldsymbol{G}$ is Pnor if and only if $m_{11} = 0$ and m_{10} is even.
(b) $\boldsymbol{G}$ is Pmis if and only if $m_{11} = 0$ and
$[(m_{00} = 0$ and m_{10} is odd) or ($m_{00} > 0$ and m_{10} is even)].

Note however that this only refers to the obliged game when one is about to choose an available game in which to oblige an opponent to play. If one is already obliged to finish one of the component games, there will be usually many other *Pnor* or *Pmis* positions.

Example 1. Obliged Wythoff's Games. Wythoff's Game is played with two piles of counters. The possible moves are to remove any positive number of counters from either pile, or to remove the same number of counters from both piles. Thus a position may be represented as a pair of nonnegative integers, (n_1, n_2), and a move is given by reducing one of the two numbers by an arbitrary amount, or by reducing both numbers by the same amount. The only terminal position is $(0, 0)$. Another description of Wythoff's Game, useful here, consists of a queen placed on the upper right quadrant of a rectangular board, with coordinates (n_1, n_2), $n_1 \geq 0$ and $n_2 \geq 0$. The queen must move down, or to the left, or diagonally down to the left.

Playing the disjunctive sum of Wythoff's games involves computing the Sprague-Grundy function of the positions, $g(n_1, n_2)$. Although there is no known formula expressing the value of $g(n_1, n_2)$, it is easy to compute numerically. See Winning Ways [1], Chapter 3. Nevertheless, there is a simple formula, found by Wythoff [3], for finding all *Pnor* positions, namely, $(\lfloor n\tau \rfloor, \lfloor n\tau^2 \rfloor)$, for $n = 0, 1, 2, \ldots$, and the symmetric values, $(\lfloor n\tau^2 \rfloor, \lfloor n\tau \rfloor)$, where $\tau = (1 + \sqrt{5})/2$ is the golden number. The *Pnor* positions start out $(0, 0), (1, 2), (2, 1), (3, 5), (5, 3), (4, 7), (7, 4), \cdots$.

It is easy to see that the *Pmis* positions are the same as the *Pnor* positions except for the first three. The *Pmis* positions are $(0, 1), (1, 0), (2, 2), (3, 5), (5, 3), (4, 7), (7, 4), \cdots$. See [1] Chapter 13. Therefore, the obliged values of the positions are as follows:

$$Val(n_1, n_2) = \begin{cases} 01 & \text{for } (n_1, n_2) = (0,0) \text{ or } (1,2) \text{ or } (2,1) \\ 10 & \text{for } (n_1, n_2) = (0,1) \text{ or } (1,0) \text{ or } (2,2) \\ 00 & \text{for } (n_1, n_2) = (3, 5), (5, 3), (4, 7), (7, 4), \cdots \\ 11 & \text{for all other values of } (n_1, n_2). \end{cases}$$

This is illustrated in Figure 1, where the numerous empty squares are to be filled with the value 11.

7					00			
6								
5				00				
4								00
3						00		
2		01	10					
1	10		01					
0	01	10						
	0	1	2	3	4	5	6	7

Fig. 1

7								
6								
5								
4		Q						
3						Q		
2			Q					
1	Q		Q					
0								
	0	1	2	3	4	5	6	7

Fig. 2

As an illustration, consider the set of queens as in Figure 2. We are to consider this as an obliged sum of five games, one for each queen. For the queens at positions (0,1), (1,4), (2,1), (2,2), and (5,3), the values, as seen from Figure 1, are respectively: 10, 11, 01, 10, and 00. Since position (1,4) has value 11, we know that the player to move may win either the normal version or the misère version of the obliged sum by starting play in the component game with the queen on (1,4). To decide whether he should win the normal version or the misère version of this game, we must find the value of the obliged game without the queen on (1,4). From Theorem 1, this value is 00. So whether he should win the normal version or the misère version of the obliged game, he must win the normal version of (1,4), so that his opponent is required to move in the resulting position. The unique optimal move from (1,4) is to (1,2).

If the queen with value 00 on (5,3) were not there, then the value of the obliged game after the game (1,4) is completed would be 01. So to win the obliged game in normal mode, he would have to win the the game (1,4) normally; to win the obliged game misère, he would have to win the game (1,4) misère.

Example 2. Obliged Knight Games. In the Knight Game, a knight is placed on a quarter infinite board with coordinates (i, j) with $i \geq 0$ and $j \geq 0$. The knight may make a knight move only to decrease the sum of the coordinates. That is a knight at (n_1, n_2) may move only to one of $(n_1 - 2, n_2 + 1)$, $(n_1 - 2, n_2 - 1)$, $(n_1 - 1, n_2 - 2)$ or $(n_1 + 1, n_2 - 2)$, and only if the resulting move is on the board. The game ends when the knight ends on one of the four squares. $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$.

The *Pnor* and *Pmis* positions are easy to find inductively. The values are exhibited in Figure 3. The values are symmetric about the diagonal and they also follow a repeated pattern of period 4 as indicated by the thicker lines.

As an example, suppose there are four knights at positions (3,4) (value 11), at (5,3) (value 10), at (4,9) (value 01) and at (6,4) (value 10). Since there is only one knight on

10	10	11	11	11	10	11	10	11	10	11	10
9	01	01	11	10	01	00	11	10	01	00	11
8	00	01	11	11	00	01	10	11	00	01	10
7	10	10	11	10	11	10	11	10	11	10	11
6	10	11	11	11	10	11	10	11	10	11	10
5	01	01	11	10	01	00	11	10	01	00	11
4	00	01	11	11	00	01	10	11	00	01	10
3	10	10	11	10	11	10	11	10	11	10	11
2	10	11	11	11	11	11	11	11	11	11	11
1	01	01	11	10	01	01	11	10	01	01	11
0	01	01	10	10	00	01	10	10	00	01	10
	0	1	2	3	4	5	6	7	8	9	10

Fig. 3

a square with value 11, that knight on square (3,4) must be moved first. The other three knights have no value 00 and an even number of values 10, so by Theorem 1, the value of the obliged game without the knight on (3,4) is 01. Therefore, if we are playing under the normal ending rule, we want the opponent to move first in this game. So we must win the normal version of the game starting at (3,4). The unique move to accomplish this is to move the knight on (3,4) to (1,5). Similarly, if we want to win the misère version of the obliged game, we must win the misère version of the game starting at (3,4). The unique move accomplishing this is to move the knight on (3,4) to (1,3).

3. Obliged Take-and-Break Games. In Take-and-Break games, a move in a game g may be to break it into two or more games. Well known examples include Kayles Dawson's Chess, and Grundy's Game. In the standard disjunctive sum of games, $G = g_1 + g_2 + \cdots + g_n$, a move that breaks g_1, say, into g_{11} and g_{12}, moves the game G into the disjunctive sum $G_1 = g_{11} + g_{12} + g_1 + \cdots + g_n$.

Here we consider Obliged Take and Break Games in which the new games are to be played as an obliged sum. If in $G = g_1 \oplus g_2 \oplus \cdots \oplus g_n$, a move breaks g_1 into g_{11} and g_{12}, we treat the move as if it were to $g_{11} \oplus g_{12}$ where both g_{11} and g_{12} must be completed before $g_2, \ldots, g_n$ is started. We may choose which of g_{11} and g_{12} to start with but the one started must be completed before the other is started.

Nearly all well known Take-and-Break games involve a pile of chips and a move consists of taking some chips and possibly splitting the remaining chips into two piles. In solving obliged take-and-break games, one may proceed by induction by finding the values of a position with n chips based on the values of positions with fewer than n chips.

First, we must find the value of the obliged sum of two games in terms of the values of the components. This is easily done as another Corollary of Theorem 1. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two games with respective values $Val(\boldsymbol{a})$ and $Val(\boldsymbol{b})$. Then the value of $\boldsymbol{a} \oplus \boldsymbol{b}$ is as given in tabular form in Figure 4.

$Val(\boldsymbol{a})$ \ $Val(\boldsymbol{b})$	00	01	10	11
00	00	00	11	11
01	00	01	10	11
10	11	10	01	11
11	11	11	11	11

Fig. 4. $Val(\boldsymbol{a} \oplus \boldsymbol{b})$

Suppose we are given the values of all positions with fewer than n chips in an obliged break-and-take game. To find the value of a position of n chips, we first find the values of all options of n, that is, the values of those positions into which n can be moved. For an option consisting of a single pile of size less than n, we know its value. For an option consisting of an obliged sum of two piles, we can find the value using the function displayed in Figure 4. In general, If there is an option that is *Pnor*, then n is *Nnor*; if all options of n are *Nnor*, then n is *Pnor*. That is, if one of the options has a value with first coordinate 0, then the value of n has first coordinate 1. This is just the *bitwise negation* (*NOT*) of the *bitwise and* function (*AND*). The same argument applies to the second coordinate. In symbols,

$$Val(n) = NOT\Big[AND\ \big[Val(\boldsymbol{a}) : \text{for all options, } \boldsymbol{a}, \text{ of } n\big]\Big].$$

This algorithm may be used for any game based on reducing the size of a pile of chips by taking and breaking. Here are some examples.

Example 3. Obliged Grundy's Game. In Grundy's Game, the only valid moves are those that break a pile of chips into two non-empty piles of different sizes. There are exactly two terminal positions: a pile of one chip, and a pile of two chips. Therefore in Obliged Grundy's Game, we have $Val(1) = Val(2) = 01$. A pile of three chips may be split into a pile of 1 chip and a pile of 2 chips: $3 \to 1 \oplus 2$. From Figure 4, we see that $Val(1 \oplus 2) = 01$, so $Val(3)$ is the negation, $Val(3) = 10$. The only move from a pile of 4 chips is $4 \to 1 \oplus 3$, and since $Val(1 \oplus 3) = 10$, we have $Val(4) = 01$. A pile of 5 chips has two moves: $5 \to 1 \oplus 4$ and $5 \to 2 \oplus 3$. These have values 01 and 10 respectively. The *AND* of 01 and 10 is 00, and the *NOT* of this is 11, so $Val(5) = 11$. Continuing on, we find

$n:$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\dots$
$Val(n):$	01	01	10	01	11	10	01	11	10	01	11	10	01	11	$\dots$

We see that, starting at $n = 3$, the values seem to be periodic of period 3 in the order 10, 01, 11. It is easy to check that this continues indefinitely.

Example 4. Obliged Kayles. In Kayles, one can remove one or two chips from any pile and, if it is desired and possible, split the remaining chips into two nonempty piles. In the octal notation of games found in [1], say, this game is **·77**. The only terminal position is the empty pile: $Val(0) = 01$. The values for Obliged Kayles may be computed as above. We find

$n:$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\dots$
$Val(n):$	01	10	11	11	10	01	11	11	01	10	11	11	10	01	11	11	$\dots$

Here the values are periodic of period 8 starting from $n = 0$.

Example 5. Obliged Dawson's Chess. Dawson's chess in octal notation is **·137**. This means that one chip may be taken only if it is the whole pile; two chips may be taken but the remaining pile cannot be split; Three chips may always be taken and, if desired and possible, the remaining chips may be split into two nonempty piles. The only terminal position is the empty pile. The values for Obliged Dawson's Chess are

$n:$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\dots$
$Val(n):$	01	10	10	11	01	11	10	10	01	01	11	10	11	01	01	10	$\dots$

These are periodic of period 14 starting from n=0.

Example 6. Obliged James Bond. In [1], the game of Treblecross is introduced as a one-dimensional version of tic-tac-toe on a linear board of n squares with both players using X; the first player to complete 3 X's in a row wins. This game is sometimes called James Bond because as a game under the normal play rule, it is equivalent to the game with octal representation **·007**. Here we analyze not Obliged Treblecross, but Obliged **·007**. This means that the only permissible moves require removing three chips from a pile and, if desired and possible, splitting the remaining chips into two nonempty piles. The only terminal positions are piles with zero, one or two chips. The values are

$n:$	0	1	2	3	4	5	6	7	8	9	10
$Val(n):$	01	01	01	10	10	10	11	11	01	11	11

$n:$	11	12	13	14	15	16	17	18	19	20	21
$Val(n):$	10	10	10	01	01	01	11	11	10	11	11

These values have period 22 starting from $n = 0$.

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