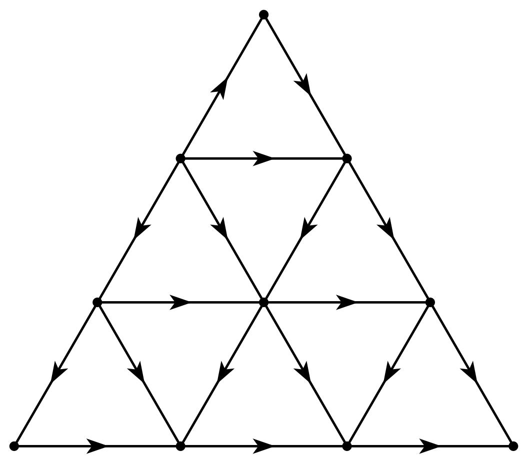
First Midterm Examination

Mathematics 167, Game Theory

Ferguson | Fri. Apr. 29, 2005

1. Find all best initial moves, if any, in the game of nim with three piles of sizes 13, 17 and 26. Find an optimal move in the misère version of the game.

2. Find the Sprague-Grundy function of the following graph. Which positions are P-positions?



3. Consider the combinatorial game in which a move consists of either (1) removing one, two or three chips from any pile, or (2) splitting a pile of size $n \geq 2$ into two piles of sizes 1 and $n-1$.
(a) Find the Sprague-Grundy function for this game.
(b) Suppose in this game there are three piles of sizes 3, 5 and 7. Show this is an N-position and find all winning moves.

4. Find the value and optimal strategies of the following games.

(a) $\begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -4 & 5 & 4 \\ 0 & 0 & 2 & 1 \\ 4 & 6 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -3 & 0 \\ 2 & 3 & 1 \\ -2 & 5 & -1 \end{pmatrix}$.

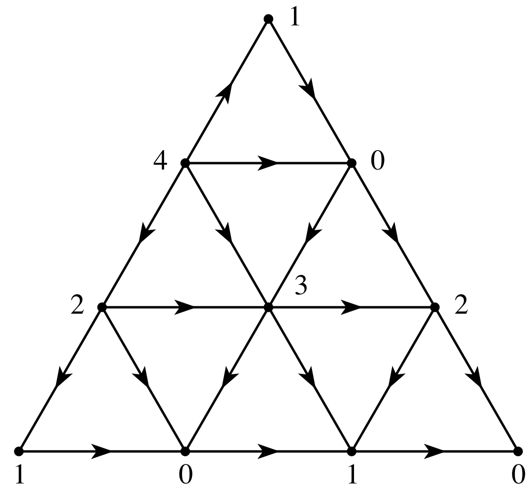
5. Show that the game with matrix $\begin{pmatrix} 1 & 3 & 6 \\ 3 & 1 & 6 \\ 4 & 6 & 0 \\ 6 & 4 & 0 \end{pmatrix}$ is invariant under a non-trivial permutation of rows and columns. Reduce using invariant strategies to a 2 by 2 matrix and solve. Relate the solution back to the original game matrix.

Solution to the First Midterm Examination
Mathematics 167, Game Theory

Ferguson Fri. Apr. 29, 2005

1. $13 \oplus 17 \oplus 26 = 6$, an N-position. The unique winning move is to remove 2 from the pile of 13 leaving 11, since $11 \oplus 17 \oplus 26 = 0$. For the misẽre version of the game, the same move is optimal.

2.



The positions with Sprague-Grundy value 0 are P-positions.

3. (a)

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14...
$g(x)$	0	1	2	4	0	3	1	2	0	3	1	2	0	3	1...

For $k \geq 1$, $g(4k) = 0$, $g(4k+1) = 3$, $g(4k+2) = 1$ and $g(4k+3) = 2$.

(b) $g(3) \oplus g(5) \oplus g(7) = 4 \oplus 3 \oplus 2 = 5$, so this is an N-position. Removing 2 chips from the pile of 3 leaving 1 gives a P-position since $g(1) \oplus g(5) \oplus g(7) = 1 \oplus 3 \oplus 2 = 0$.

4. (a) (5/6,1/6) is optimal for I, (1/3,2/3) is optimal for II and the value is 5/3.

(b) Col 3 is dominated by col 4. Row 2 is dominated by $\frac{1}{2}$row1 + $\frac{1}{2}$row 3. Col 1 is dominated by $\frac{1}{2}$col 2 + $\frac{1}{2}$col 4. (3/7,0,4/7) is optimal for I, (0,2/7,0,5/7) is optimal for II and the value is 12/7.

(c) There is a saddle point at (row 2, col 3). Row 2 is optimal for I, col 3 is optimal for II and the value is 1.

5. If the first two columns are interchanged and then the first two rows are interchanged and the last two rows are interchanged, the matrix remains the same. The game is invariant under these motions. Therefore Player II may restrict her choices to $\{12^*, 3\}$ where 12^* represents the strategy that gives probability 1/2 to col 1 and 1/2 to col 2.

Similarly Player I may restrict his choices to $\{12^*, 34^*\}$. The game matrix reduces to

$$\begin{array}{cc} & \begin{array}{cc} 12^* & 3 \end{array} \\ \begin{array}{c} 12^* \\ 34^* \end{array} & \left(\begin{array}{cc} 2 & 6 \\ 5 & 0 \end{array}\right) \end{array}$$

The optimal strategies are (5/9,4/9) for Player I and (2/3,1/3) for Player II. The value is 30/9. In the original matrix, the optimal strategies sre (5/18,5/18,2/9,2/9) for I and (1/3,1/3,1/3) for II. The value is 30/9.