Solutions to Exercise Set 9.

20.5.(a) $E_\theta(X) = (1-\theta) + 2\theta = 1+\theta$, so the method of moments estimator of θ is $\hat\theta_n = \overline{X}_n - 1$.

(b) The log likelihood is $\ell_n(\theta) = \sum \log[(1-\theta)e^{-x_i} + \theta(1/2)e^{-x_i/2}]$, so the likelihood equation is

$$\ell_n'(\theta) = \sum \frac{-e^{-x_i/2} + (1/2)}{(1-\theta)e^{-x_i/2} + \theta(1/2)} = 0.$$

One may improve $\hat\theta_n$ by the Newton method formula $\tilde\theta_n = \hat\theta_n - \ell_n'(\hat\theta_n)/\ell_n''(\hat\theta_n)$, where

$$\ell_n''(\theta) = -\sum \frac{(e^{-x_i/2} - (1/2))^2}{[(1-\theta)e^{-x_i/2} + \theta(1/2)]^2}.$$

22.1. (a) The likelihood function is $L = \prod(1/\sqrt{2\pi}\sigma)e^{-(y_i-\alpha-\beta x_i)^2/2\sigma^2}$. The MLE's under the general hypothesis, H, are $\hat\beta = s_{xy}/s_x^2$, $\hat\alpha = \overline{Y} - \hat\beta\overline{X}$, and $\hat\sigma^2 = \frac{1}{n}\sum(Y_i - \hat\alpha - \hat\beta x_i)^2$. We have $\sup_H L = (1/\sqrt{2\pi}\hat\sigma)^n e^{-n/2}$. Under H_0, the MLE's are $\tilde\alpha = \tilde\beta = \sum Y_i(x_i+1)/\sum(x_i+1)^2$, and $\tilde\sigma^2 = \frac{1}{n}\sum(Y_i - \tilde\alpha(x_i+1))^2$. We have $\sup_{H_0} L = (1/\sqrt{2\pi}\tilde\sigma)^n e^{-n/2}$. The likelihood ratio statistic is $\Lambda = (\hat\sigma/\tilde\sigma)^n$. The likelihood ratio test rejects H_0 if Λ is too small. This is equivalent to the test that rejects H_0 if $\tilde\sigma^2/\hat\sigma^2$ is too large, or if $F = (\tilde\sigma^2 - \hat\sigma^2)/(\frac{1}{n-2}\hat\sigma^2)$ is too large. This is the usual F-test of H_0. Under H_0, the statistic F has an F-distribution with 1 and $n-2$ degrees of freedom.

(b) If σ^2 is given, the estimates of α and β are the same as in (a). Here we have $\sup_H L = (\frac{1}{\sqrt{2\pi}\sigma})^n e^{-n\hat\sigma^2/2\sigma^2}$, and $\sup_{H_0} L = (\sqrt{2\pi}\sigma)^n e^{-n\tilde\sigma^2/2\sigma^2}$. The likelihood ratio statistic is $\Lambda = \exp\{-\frac{n}{2\sigma^2}(\tilde\sigma^2 - \hat\sigma^2)\}$. The likelihood ratio test rejects H_0 if Λ is too small. This is equivalent to the test that rejects H_0 if $\tilde\sigma^2 - \hat\sigma^2$ is too large. This statistic is the numerator of the F above. It has a chi-square distribution with 1 degree of freedom.

22.5.(a) The likelihood function is $L(\mu,\theta) = e^{-n\mu}\mu^{\Sigma X_i}e^{-n\theta}\theta^{\Sigma Y_i}/(\prod x_i!y_i!)$. The maximum likelihood estimates under the general hypothesis are $\hat\mu_n = \overline{X}_n$ and $\hat\theta_n = \overline{Y}_n$. To find the maximum likelihood estimates under H_0, we replace μ by θ^2 in $\log L$ and take a derivative with respect to θ.

$$\frac{\partial}{\partial\theta}\log L_n = -2n\theta + \frac{2\sum X_i}{\theta} - n + \frac{\sum Y_i}{\theta} = 0$$

Solving the quadratic for θ gives $\tilde\theta_n = (-1 + \sqrt{1 + 8(2\overline{X} + \overline{Y})})/4$ as the MLE of θ under H_0. The likelihood ratio statistic then becomes

$$\lambda_n = \frac{L(\tilde\theta_n^2, \tilde\theta_n)}{L(\hat\mu_n, \hat\theta_n)} = \frac{e^{-n\tilde\theta_n^2 - n\tilde\theta_n}\tilde\theta_n^{[2n\bar X + n\bar Y]}}{e^{-n\hat\mu_n - n\hat\theta_n}\hat\mu_n^{n\bar X}\hat\theta_n^{n\bar Y}}$$

(b) $-2\log\lambda_n \xrightarrow{\mathcal{L}} \chi_1^2$ as $n\to\infty$.