Solutions to Exercise Set 6.

12.4. (a) Write $Z-\mathrm{E}Z=\sum_1^N(z_j-\bar z_N)a(R_j)$ and $T-\mathrm{E}T$ similarly. Then, $\mathrm{Cov}(Z,T)=\mathrm{Cov}(Z-\mathrm{E}Z,T-\mathrm{E}T)=\sum_1^N\sum_1^N(z_i-\bar z_N)(t_j-\bar t_N)\mathrm{Cov}(a(R_i),b(R_j))$. There are only two values of $\mathrm{Cov}(a(R_i),b(R_j))$, namely when $i=j$ and when $i\neq j$. Moreover,

$$\mathrm{Cov}(a(R_1),b(R_1))=\frac{1}{N}\sum_{i=1}^N(a(i)-\bar a_n)(b(i)-\bar b_N)=\sigma_{ab},$$

and

$$\begin{aligned}\mathrm{Cov}(a(R_1),b(R_2))&=\frac{1}{N(N-1)}\sum\sum_{i\neq j}(a(i)-\bar a_n)(b(j)-\bar b_N)\\&=\frac{1}{N(N-1)}\left[-\sum(a(i)-\bar a_N)(b(i)-\bar b_N)\right]=-\frac{\sigma_{ab}}{N-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Cov}(Z,T)&=\sum_{i=1}^N(z_i-\bar z_N)(t_i-\bar t_N)\sigma_{ab}+\sum\sum_{i\neq j}(z_i-\bar z_N)(t_j-\bar t_N)(-\frac{\sigma_{ab}}{N-1})\\&=N\sigma_{zt}\sigma_{ab}-\frac{\sigma_{ab}}{N-1}\left[-\sum_{i=1}^N(z_i-\bar z_N)(t_i-\bar t_N)\right]\\&=N\sigma_{zt}\sigma_{ab}+\frac{N}{N-1}\sigma_{zt}\sigma_{ab}=\frac{N^2}{N-1}\sigma_{zt}\sigma_{ab}.\end{aligned}$$

(b) Let $S_N=\alpha Z_N/N^2+\beta T_N/N^3$. We show S_N is asymptotically normal and apply Exercise 3.2 of the text. We have $\bar z_n=\bar t_n=\bar b_n=(N+1)/2$, $\bar a_N=m/N$, $\sigma_z^2=\sigma_{zt}=\sigma_t^2=\sigma_b^2=(N^2-1)/12$, $\sigma_a^2=mn/N^2$ and $\sigma_{ab}=-mn/(2N)$, where $n=N-m$. Note that $S_N=\sum z_jc(R_j)$, where $c(R_j)=\alpha a(R_j)/N^2+\beta b(R_j)/N^3$. We have

$$\begin{aligned}(c(j)-\bar c_N)^2&=\left(\frac{\alpha}{N^2}(a(j)-\bar a_N)+\frac{\beta}{N^3}(b(j)-\bar b_N)\right)^2\\&=\frac{\alpha^2}{N^4}(a(j)-\bar a_N)^2+\frac{2\alpha\beta}{N^5}(a(j)-\bar a_N)(b(j)-\bar b_N)+\frac{\beta^2}{N^6}(b(j)-\bar b_N)^2\end{aligned}$$

and

$$\begin{aligned}\sigma_c^2&=\frac{\alpha^2}{N^4}\sigma_a^2+\frac{2\alpha\beta}{N^5}\sigma_{ab}+\frac{\beta^2}{N^6}\sigma_b^2\\&\sim\frac{\alpha^2}{N^4}r(1-r)-\frac{\alpha\beta}{N^4}r(1-r)+\frac{\beta^2}{12N^4}\end{aligned}$$

From this we see that both $\max_j (c(j) - \bar{c}_N)^2$ and σ_c^2 tend to a constant when divided by N^4. Thus the ratio stays bounded. We already know that $\max_j (z_j - \bar{z}_N)^2/\sigma_z^2$ stays bounded, so we have $\delta_N \to 0$. This implies that the normalized version of S_N is asymptotically normal. From

$$\begin{aligned}\mathrm{Var}(Z_N) &= \frac{N^2}{N-1}\sigma_z^2\sigma_a^2 &&\sim \frac{N^3}{12}r(1-r)\\ \mathrm{Var}(T_N) &= \frac{N^2}{N-1}\sigma_t^2\sigma_b^2 &&\sim \frac{N^5}{12^2}\\ \mathrm{Cov}(Z_N, T_N) &= \frac{N^2}{N-1}\sigma_{zt}\sigma_{ab} &&\sim -\frac{N^4}{24}r(1-r)\end{aligned}$$

we find that the variance is

$$\begin{aligned}\mathrm{Var}(S_N) &= \frac{\alpha^2}{N^4}\mathrm{Var}(Z_N) + \frac{2\alpha\beta}{N^5}\mathrm{Cov}(Z_N, T_N) + \frac{\beta^2}{N^6}\mathrm{Var}(T_N)\\ &\sim \frac{1}{N}(\alpha^2\frac{r(1-r)}{12} - 2\alpha\beta\frac{r(1-r)}{24} + \beta^2\frac{1}{144}).\end{aligned}$$

Hence,

$$\begin{aligned}\sqrt{N}(S_N - \mathrm{E}S_N) &= \sqrt{N}(\alpha(\frac{Z}{N^2} - \frac{r}{2}) + \beta(\frac{T_N}{N^3} - \frac{1}{4}))\\ &\xrightarrow{\mathcal{L}} \mathcal{N}(0, \alpha^2\frac{r(1-r)}{12} - 2\alpha\beta\frac{r(1-r)}{24} + \beta^2\frac{1}{144})\end{aligned}$$

from which the result follows.

13.2. (a) The pth quantile of F is $u_i + \theta$. So for $Z_i = X_{(\lceil np_i \rceil)} - u_i$, we have from the Corollary of Chapter 13, $\sqrt{n}(\boldsymbol{Z} - \theta\mathbf{1}) \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \Sigma)$, where $\Sigma = (\sigma_{ij})$ and for $i \le j$, $\sigma_{ij} = p_i(1-p_j)/(f(u_i)f(u_j))$.

(b) Using Lagrange mulipliers, write $\varphi(\boldsymbol{a}) = \boldsymbol{a}^T\Sigma\boldsymbol{a} - \lambda\mathbf{1}^T\boldsymbol{a}$. Then $\dot{\varphi}(\boldsymbol{a}) = 2\Sigma\boldsymbol{a} - \lambda\mathbf{1} = 0$ has solution $\boldsymbol{a} = \lambda\Sigma^{-1}\mathbf{1}/2$. To find λ, $1 = \mathbf{1}^T\boldsymbol{a} = \lambda\mathbf{1}^T\Sigma^{-1}\mathbf{1}/2$, so that $\lambda = 2/\,\mathbf{1}^T\Sigma^{-1}\mathbf{1}$. Therefore, $\boldsymbol{a} = \Sigma^{-1}\mathbf{1}\,/\,\mathbf{1}^T\Sigma^{-1}\mathbf{1}$.

(c) Let $g_i = f(u_i)$ for $i = 1, \ldots, k$, and let $p_0 = 0$ and $p_{k+1} = 1$. Then $\Sigma^{-1} = \left(\sigma^{ij}\right)$, where

$$\sigma^{ij} = \begin{cases} \frac{(p_{i+1}-p_{i-1})g_i^2}{(p_{i+1}-p_i)(p_i-p_{i-1})} & \text{if } j = i\\ -\frac{g_i g_{i+1}}{(p_{i+1}-p_i)} & \text{if } j = i+1\\ -\frac{g_i g_{i-1}}{(p_i-p_{i-1})} & \text{if } j = i-1\\ 0 & \text{otherwise.}\end{cases}$$

(d) For the uniform distribution, $g_i = 1$ for all i, so the vector $\Sigma^{-1}\mathbf{1}$ is the transpose of $(1/p_1, 0, \ldots, 0, 1/(1-p_k))$, and $\hat{\theta} = (Z_1/p_1 + Z_k/(1-p_k))/(1/p_1 + 1/(1-p_k))$.