Stat 200C, Spring 2009 Ferguson

Solutions to Exercise Set 1.

1.4. (a) The density of X is $f_X(x) = \alpha x^{-\alpha-1} I(x > 1)$. Therefore the expected value of X is $E(X) = \alpha \int_1^\infty x^{-\alpha}\,dx = \alpha/(\alpha - 1)$. The second moment is $E(X^2) = \alpha \int_1^\infty x^{-\alpha+1}\,dx = \alpha/(\alpha-2)$. The variance is therefore $E(X^2) - (E(X))^2 = \alpha/((\alpha-1)^2(\alpha-2))$.

(b) If $Y = (\alpha - 1)X - \alpha$, Then for $y > -1$,

$$
\begin{aligned}
P(Y \le y) &= P((\alpha-1)X - \alpha \le y) = P(X \le \frac{\alpha + y}{\alpha - 1}) \\
&= 1 - \left(\frac{\alpha+y}{\alpha-1}\right)^{-\alpha} = 1 - \left(1 + \frac{y+1}{\alpha-1}\right)^{-\alpha} \\
&\to 1 - e^{-(y+1)}
\end{aligned}
$$

as $\alpha \to \infty$. This is the distribution function of the exponential distribution of mean 1, centered at its mean.

2.1. (a) The density of Y is

$$
f(y) = \frac{|\gamma|\alpha^\alpha}{\Gamma(\alpha)} \exp\{-\alpha e^{\gamma(y-\theta)} + \alpha\gamma(y-\theta)\} \quad \text{for} \quad -\infty < y < \infty.
$$

θ is a location parameter since this depends on y and θ only through the difference $y - \theta$.

(b) $\mathrm{E}Y = (1/\gamma)\mathrm{E}\log(X)$. Make the change of variable $U = X/\beta$ to find

$$
\begin{aligned}
\mathrm{E}Y &= (1/\gamma)\int_0^\infty (\log(u) + \log(\beta))e^{-u}u^{\alpha-1}\,du/\Gamma(\alpha) = (1/\gamma)[\psi(\alpha) + \log(\beta)] \\
&= \theta + (1/\gamma)[\psi(\alpha) - \log(\alpha)].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathrm{E}Y^2 &= (1/\gamma^2)\int_0^\infty (\log(u) + \log(\beta))^2 e^{-u}u^{\alpha-1}\,du/\Gamma(\alpha) \\
&= (1/\gamma^2)[\Gamma''(\alpha)/\Gamma(\alpha) + 2\log(\beta)\Gamma'(\alpha) + (\log(\beta))^2].
\end{aligned}
$$

Since $\psi'(\alpha) = (d/d\alpha)\Gamma'(\alpha)/\Gamma(\alpha) = (\Gamma''(\alpha)/\Gamma(\alpha)) - (\psi(\alpha))^2$, we have $\mathrm{E}Y^2 = (1/\gamma^2)[\psi'(\alpha) + \psi(\alpha)^2 + 2\log(\beta)\psi(\alpha) + (\log(\beta)^2]$, and $\mathrm{Var}Y = (1/\gamma^2)\psi'(\alpha) = \sigma^2$.

(c) If we let $\gamma \to 0$ with θ and σ fixed, then $\alpha \to \infty$ since $\gamma^2\sigma^2 = \psi'(\alpha)$. From Stirling's formula, $\Gamma(\alpha) = (1/\alpha)(\Gamma(\alpha+1) \sim (1/\alpha)\alpha^\alpha e^{-\alpha}\sqrt{2\pi\alpha}$, we have

$$
f_Y(y) \sim \frac{|\gamma|\sqrt{\alpha}}{\sqrt{2\pi}} \exp\{-\alpha[e^{\gamma(y-\theta)} - \gamma(y-\theta) - 1]\}.
$$

Since $\alpha\psi'(\alpha) \to 1$, we have $\gamma^2\alpha \to 1/\sigma^2$. If the term $e^{\gamma(y-\theta)}$ is expanded in a power series, the first two terms cancel and the limit as $\gamma \to 0$ becomes $f_Y(y) \to (1/\sqrt{2\pi}\sigma)\exp\{-(y-\theta)^2/(2\sigma^2)\}$.