Solutions to Exercises 5.9.3 through 5.9.9.

5.9.3.
$$\begin{aligned} S^2 &= \sum\sum(X_{ij}-\xi-\mu_i-\eta_j)^2 \\ &= \sum\sum[(X_{ij}-\overline{X}_{i.}-\overline{X}_{.j}-\overline{X}_{..})+(\overline{X}_{i.}-\overline{X}_{..}-\mu_i)+(\overline{X}_{.j}-\overline{X}_{..}-\eta_j)-(\overline{X}_{..}-\xi)]^2 \\ &= \sum\sum(X_{ij}-\overline{X}_{i.}-\overline{X}_{.j}-\overline{X}_{..})^2+\sum\sum(\overline{X}_{i.}-\overline{X}_{..}-\mu_i)^2 \\ &+\sum\sum(\overline{X}_{.j}-\overline{X}_{..}-\eta_j)^2+\sum\sum(\overline{X}_{..}-\xi)^2 \\ &+ \text{ six cross product terms.} \end{aligned}$$

We must show that all cross product terms are zero. This is illustrated for the first term,

$$2\sum\sum(X_{ij}-\overline{X}_{i.}-\overline{X}_{.j}-\overline{X}_{..})(\overline{X}_{i.}-\overline{X}_{..}-\mu_i)=\sum_{i=1}^{I}(\overline{X}_{i.}-\overline{X}_{..}-\mu_i)\sum_{j=1}^{J}(X_{ij}-\overline{X}_{i.}-\overline{X}_{.j}-\overline{X}_{..})=0$$

The inside summation over j is zero for all i because $\sum_j X_{ij}=J\overline{X}_{i.}$ and $\sum_j \overline{X}_{.j}=J\overline{X}_{..}$.

5.9.4. (a) We expand S^2 as follows.

$$\begin{aligned} S^2 &= \sum\sum\sum(X_{ijk}-\xi-\mu_i-\eta_j-\delta_{ij})^2 \\ &= \sum\sum\sum[(X_{ijk}-\overline{X}_{ij.})-(\overline{X}_{ij.}-\xi-\mu_i-\eta_j-\delta_{ij})]^2 \\ &= \sum\sum\sum(X_{ijk}-\overline{X}_{ij.})^2+\sum\sum\sum(\overline{X}_{ij.}-\xi-\mu_i-\eta_j-\delta_{ij})^2 \\ &+2\sum_i\sum_j(\overline{X}_{ij.}-\xi-\mu_i-\eta_j-\delta_{ij})\sum_k(X_{ijk}-\overline{X}_{ij.}) \end{aligned}$$

The last term is zero since $\sum_k X_{ijk}=K\overline{X}_{ij.}$ for all i and j. The middle term may be expanded into a sum of squares using the identity (5.128) (or Exercise 3) with X_{ij} replaced by $\overline{X}_{ij.}-\delta_{i,j}$.

(b) Using this identity, we find

$$\min_H S^2=\sum\sum\sum(X_{ijk}-\overline{X}_{ij.})^2.$$

Similarly, putting $\mu_i=0$ in the formula for part (a), we find

$$\min_{H_0} S^2=\sum\sum\sum(X_{ijk}-\overline{X}_{ij.})^2+\sum\sum\sum(\overline{X}_{i..}-\overline{X}_{...})^2.$$

The best invariant test rejects H_0 if

$$F=\frac{(\min_{H_0}S^2-\min_H S^2)/r}{\min_H S^2/(n-k)}=\frac{\sum\sum\sum(\overline{X}_{i..}-\overline{X}_{...})^2/(I-1)}{\sum\sum\sum(X_{ijk}-\overline{X}_{ij.})^2/(IJ(K-1))}$$

is too large. Under H_0, this statistic has an F-distribution with $I-1$ and $IJ(K-1)$ degrees of freedom. Under the general hypothesis, H, it has a noncentral F-distribution with $I-1$ and $IJ(K-1)$ degrees of freedom and noncentrality parameter γ^2, where γ^2 is computed as the numerator sum of squares divided by σ^2, with each X_{ijk} replaced by its expectation. This results in replacing $\overline{X}_{i..}$ by $\mu_i+\xi$ and $\overline{X}_{...}$ by ξ. We find

$$\gamma^2=\frac{1}{\sigma^2}\sum\sum\sum\mu_i^2=\frac{1}{\sigma^2}JK\sum_i\mu_i^2.$$

(c) If we put $\delta_{ij}=0$ in the identity of part (a), we find $\min_{H_0}S^2=\sum\sum\sum(X_{ijk}-\overline{X}_{ij.})^2+\sum\sum\sum(\overline{X}_{ij.}-\overline{X}_{i..}-\overline{X}_{.j.}+\overline{X}_{...})^2$ so the best invariant test of no interaction effect rejects H_0 if

$$F=\frac{\sum\sum\sum(\overline{X}_{ij.}-\overline{X}_{i..}-\overline{X}_{.j.}+\overline{X}_{...})^2/((I-1)(J-1))}{\sum\sum\sum(X_{ijk}-\overline{X}_{ij.})^2/(IJ(K-1))}$$

is too large. Under the general hypothesis, this has a noncentral F-distribution with $(I-1)(J-1)$ and $IJ(K-1)$ degrees of freedom and noncentrality parameter

$$\gamma^2 = \frac{1}{\sigma^2}\sum\sum\sum \delta_{ij}^2.$$

5.9.5. (a) $S^2 = \sum(X_i - \beta_0 - \beta_1 z_i)^2$. We may find the least squares estimates of β_0 and β_1 by equating the partial derivatives of S^2 to zero.

$$\frac{\partial S^2}{\partial \beta_0} = -2\sum(X_i - \beta_0 - \beta_1 z_i) = -2(\sum X_1 - n\beta_0) = 0$$

since $\sum z_i = 0$, and

$$\frac{\partial S^2}{\partial \beta_1} = -2\sum(X_i - \beta_0 - \beta_1 z_i)z_i = -2(\sum X_i z_i - \beta_1 \sum z_i^2) = 0.$$

This gives $\hat\beta_0 = (1/n)\sum X_i$ and $\hat\beta_1 = \sum X_i z_i / \sum z_i^2$ as the least squares estimates.

(b) Under H_0, $S^2 = \sum(X_i - \beta_0)^2$ is minimized at $\hat{\hat\beta}_0 = (1/n)\sum X_i = \hat\beta_0$.

(c) The UMP invariant test rejects H_0 if

$$F = \frac{\sum(\hat{\hat\beta}_0 - \hat\beta_0 - \hat\beta_1 z_i)^2/1}{\sum(X_i - \hat\beta_0 - \hat\beta_1 z_i)^2/(n-2)} = \frac{(n-2)\hat\beta_1^2}{\sum(X_i - \hat\beta_0 - \hat\beta_1 z_i)^2}$$

is too large. Under H, F has a noncentral F-distribution with 1 and $n-2$ degrees of freedom and noncentrality parameter, $\gamma^2 = (1/\sigma^2)(\sum(\beta_0 + \beta_1 z_i)z_i/\sum z_i^2)^2 = (1/\sigma^2)\beta_1^2$.

5.9.6. (a) The least squares estimates are the values of α, β and η_j that minimize $S^2 = \sum\sum(X_{ij} - \alpha - \beta z_i - \eta_j)^2$ subject to $\sum_j \eta_j = 0$. We use Lagrange multipliers. The Lagrangian is $L = S^2 + \lambda\sum\eta_j$.

$$\frac{\partial L}{\partial \alpha} = -2\sum\sum(X_{ij} - \alpha - \beta z_i - \eta_j) = -2(\sum\sum X_{ij} - IJ\alpha) = 0$$

using $\sum z_i = 0$ and $\eta_j = 0$. This gives $\hat\alpha = \overline{X}_{..}$.

$$\frac{\partial L}{\partial \beta} = -2\sum\sum(X_{ij} - \alpha - \beta z_i - \eta_j)z_i = -2(\sum\sum X_{ij}z_i - IJ\beta) = 0$$

using $\sum z_i^2 = 1$. This gives $\hat\beta = (1/I)\sum z_i \overline{X}_{i.}$.

$$\frac{\partial L}{\partial \eta_j} = -2\sum_{i=1}^{I}(X_{ij} - \alpha - \beta z_i - \eta_j) + \lambda = -2I(\overline{X}_{.j} - \alpha - \eta_j) + \lambda = 0.$$

We may find the Lagrange multiplier, λ, by summing over j. This gives $\lambda = 2I(\overline{X}_{..})$. Substituting this value into the equation gives $\hat\eta_j = \overline{X}_{.j} - \overline{X}_{..}$ as the least squares estimate of η_j. Under H_0 the least squares estimates are easily found to be $\hat{\hat\alpha} = \overline{X}_{..} = \hat\alpha$ and $\hat{\hat\beta} = (1/I)\sum z_i \overline{X}_{i.} = \hat\beta$. The UMP invariant test of H_0 rejects H_0 if

$$F = \frac{\sum\sum\hat\eta_j^2/(J-1)}{\sum\sum(X_{ij} - \hat\alpha - \hat\beta z_i - \eta_j)^2/(IJ - J - 1)}$$

is too large.

(b) Under the general hypothesis, F has a noncentral F-distribution with $J-1$ and $IJ-J-1$ degrees of freedom and noncentrality parameter, $\gamma^2 = (1/\sigma^2)\sum\sum\eta_j^2$.

5.9.7. Under the general linear hypothesis, (5.115), the log likelihood function of θ and σ is

$$\log f(\mathbf{x}|\theta,\sigma) = -n\log(\sqrt{2\pi}\sigma) - \frac{1}{2\sigma^2}(\mathbf{x}-\mathbf{A}\theta)^T(\mathbf{x}-\mathbf{A}\theta).$$

For each fixed σ, the maximum of $\log f$ over θ occurs at any value of θ that minimizes $(\mathbf{x}-\mathbf{A}\theta)^T(\mathbf{x}-\mathbf{A}\theta) = (\mathbf{x}-\xi)^T(\mathbf{x}-\xi)$. Since such values are independent of σ, any maximum likelihood estimate of θ is also a least squares estimate of θ and conversely. Therefore, if $\hat{\theta}$ denotes a maximum likelihood estimate, $\hat{\xi} = \mathbf{A}\hat{\theta}$ is both the maximum likelihood estimate and the least squares estimate of ξ.

5.9.8. We have $r = I-1$, $k = I$ and $n = n_1 + \cdots + n_I$. The distribution of the F-statistic under the general hypothesis is noncentral F-distribution, $F_{r,n-k}(\gamma^2) = F_{I-1,n-I}(\gamma^2)$. To find γ^2, we use (5.126). The numerator sum of squares is $\sum_i\sum_j(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot})^2 = \sum_i n_i(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot})^2$. Replacing each X_{ij} in this by its expectation θ_i, we obtain $\gamma^2 = (1/\sigma^2)\sum_i n_i(\theta_i - \bar{\theta})^2$.

5.9.9. (a) $S^2 = \sum\sum\sum(X_{ijk} - \xi - \lambda_i - \mu_j - \eta_k)^2 = \sum\sum\sum(X_{ijk} - \overline{X}_{i..} - \overline{X}_{.j.} - \overline{X}_{..k} + 2\overline{X}_{...})^2 +$
$+\sum\sum\sum(\overline{X}_{i..} - \overline{X}_{...} - \lambda_i)^2 + \sum\sum\sum(\overline{X}_{.j.} - \overline{X}_{...} - \mu_j)^2 + \sum\sum\sum(\overline{X}_{..k} - \overline{X}_{...} - \eta_i)^2 + \sum\sum\sum(\overline{X}_{...} - \xi)^2$.

(b) Under H, the least squares estimates are $\hat{\xi} = \overline{X}_{...}$, $\hat{\lambda}_i = \overline{X}_{i..} - \overline{X}_{...}$, $\hat{\mu}_j = \overline{X}_{.j.} - \overline{X}_{...}$, and $\hat{\eta}_k = \overline{X}_{..k} - \overline{X}_{...}$. Under H_0, they are the same except that $\hat{\hat{\lambda}}_i = 0$. So the best invariant test of H_0 rejects H_0 when

$$F = \frac{\sum\sum\sum(\overline{X}_{i..} - \overline{X}_{...})^2/(I-1)}{\sum\sum\sum(X_{ijk} - \overline{X}_{i..} - \overline{X}_{.j.} - \overline{X}_{..k} + 2\overline{X}_{...})^2/(IJK - I - J - K + 2)}$$

is too large. Under H, this has a noncentral F-distribution with $I-1$ and $IJK - I - J - K + 2$ degrees of freedom and noncentrality parameter $\gamma^2 = (1/\sigma^2)\sum\sum\sum\lambda_i^2$.