Solutions to the Exercises of Section 4.8.

4.8.1. We have $\bar{g}_\phi(F) = F(\phi^{-1})$ and $\tilde{g}_\phi(\hat{F}) = \hat{F}(\phi^{-1})$. We are to show $L(\bar{g}_\phi(F), \tilde{g}_\phi(\hat{F})) = L(F, \hat{F})$. For the loss of (4.63),

$$L(\bar{g}_\phi(F), \tilde{g}_\phi(\hat{F})) = \sup_x |F(\phi^{-1}(x)) - \hat{F}(\phi^{-1}(x))| = \sup_y |F(y) - \hat{F}(y)| = L(F, \hat{F})$$

since ϕ^{-1} is one-to-one and onto. For the loss of (4.68),

$$\begin{aligned} L(\bar{g}_\phi(F), \tilde{g}_\phi(\hat{F})) &= \int (F(\phi^{-1}(x)) - \hat{F}(\phi^{-1}(x)))^2 h(F(\phi^{-1}(x)))\, dF(\phi^{-1}(x)) \\ &= \int (F(y) - \hat{F}(y))^2 h(F(y))\, dF(y) = L(F, \hat{F}) \end{aligned}$$

using the change of variable $y = \phi^{-1}(x)$.

4.8.2. Equation (4.6) here reduces to

$$\hat{F}_{\phi(y_1),\ldots,\phi(y_n)}(\phi(x)) = \hat{F}_{y_1,\ldots,y_n}(x),$$

for all continuous strictly increasing functions ϕ from the real line onto the real line, for all $y_1 < \cdots < y_n$ and for all x. We now show that this implies that $\hat{F}_{y,\ldots,y_n}(x)$ is constant for x between the y_i. For suppose that z and z' are such that $y_i < z < y_{i+1}$ and $y_i < z' < y_{i+1}$. There exists a continuous strictly increasing ϕ from the real line onto the real line such that $\phi(y_i) = y_i$ for all i and $\phi(z) = z'$. (In fact, we can take $\phi(x) = x$ for $x \notin (y_i, y_{i+1})$ and ϕ linear on (y_i, z) and (z, y_{i+1}) with $\phi(z) = z'$.) Use of such a ϕ in the displayed equation gives $\hat{F}_{y_1,\ldots,y_n}(z) = \hat{F}_{y_1,\ldots,y_n}(z')$.

Thus, since $\hat{F}$ is a right continuous distribution function, we can write (4.67). Moreover, u_i does not depend on $y_1, \ldots, y_n$. For if z is fixed and $y_1 < \cdots < y_i < z < y_{i+1} < \cdots < y_n$ and $y'_1 < \cdots < y'_i < z < y'_{i+1} < \cdots < y'_n$, there exists a continuous (in fact, piecewise linear) strictly increasing function ϕ from the real line onto the real line such that $\phi(z) = z$ and $\phi(y_j) = \phi(y'_j)$ for all j. Use of this ϕ gives $F_{y'_1,\ldots,y'_n}(z) = \hat{F}_{y_1,\ldots,y_n}(z)$, completing the proof.

4.8.3. (a) The distribution of $g_\phi(Y_1, \ldots, Y_n) = (\phi(Y_1), \ldots, \phi(Y_n))$ is as the order statistics of a sample of size n from distribution function $G(x) = F(\phi^{-1}(x))$, so the distributions are invariant with $\bar{g}_\phi(F(x)) = F(\phi^{-1}(x))$. In addition, the loss is invariant with $\tilde{g}_\phi(a) = \phi(a)$, because $L(\bar{g}_\phi(F), \tilde{g}_\phi(a)) = W(\bar{g}_\phi(F(\tilde{g}_\phi(a)))) = W(F(\phi^{-1}(\phi(a))) = W(F(a)) = L(F, a)$.

(b) If δ is a behavioral rule that chooses a random variable $Z_{\mathbf{y}}$ if $\mathbf{Y} = \mathbf{y}$ is observed, then the distribution of $Z_{\mathbf{y}}$ is the same as the distribution of $\phi(Z_{\mathbf{y}})$ for all ϕ in $\Phi_{\mathbf{y}} = \{\phi : \phi(y_i) = y_i, \text{ for } i = 1, \ldots, n\}$. By the method given on page 193, $Z_{\mathbf{y}}$ must give zero mass to every interval between the y_j; that is, $Z_{\mathbf{y}}$ gives all its weight to $\{y_1, \ldots, y_n\}$. Thus, $Z_{\mathbf{y}} = \phi(Z_{\mathbf{y}})$ with probability one.

(c) We may restrict attention to the nonrandomized invariant rules. As in Exercize 4.2.3, the only nonrandomized invariant rules are $d(\mathbf{Y}) = y_j$ for some j.

(d) For $W(F(a)) = (F(a) - 1/2)^2$, the nonrandomized rule $d_i(\mathbf{y}) = y_i$ has constant risk, $R(F, d_i) = \mathrm{E}_F(F(X_{(i)}) - 1/2)^2 = \mathrm{E}(U_i - 1/2)^2$, where U_i is the ith order statistic of a sample of size n from a uniform distribution on the interval $(0, 1)$. Since U_i has a beta distribution, $\mathcal{B}e(i, n+1-i)$, we have $R(F, d_i) = \mathrm{Var}(U_i) + (\mathrm{E}U_i - 1/2)^2 = i(n+1-i)/((n+1)^2(n+2)) + (i/(n+1) - 1/2)^2$, which has a minimum at $i = (n+1)/2$ for n odd, and for $i = n/2$ or $(n+2)/2$ for n even. This is the best invariant rule.

(e) For the loss function, $W(F(a)) = F(a)$ for $F(a) > 0$ and $W(F(a)) = 1$ for $F(a) = 0$, the constant risk of d_i is $R(F, d_i) = \mathrm{E}U_i = i/(n+1)$ so the best invariant rule is d_1.