Solutions to the Exercises of Section 3.5.

3.5.1. (a) $$f(x|\lambda) = e^{-\lambda} e^{x\log(\lambda)}/x! \quad \text{for} \quad x = 0, 1, \ldots.$$

So, $k = 1$, $c(\lambda) = e^{-\lambda}$, $h(x) = 1/x!$ for $x = 0, 1, \ldots$, $\pi(\lambda) = \log(\lambda)$ and $t(x) = x$. The natural parameter space is $\Pi = \{\pi : \sum_{x=0}^{\infty} e^{\pi x}/x! < \infty\} = (-\infty, \infty)$.

(b) $$f(x|\theta) = \binom{r+x-1}{x} \exp\{x\log(\theta)\}(1-\theta)^r \quad \text{for} \quad x = 0, 1, \ldots.$$

So, $k = 1$, $c(\theta) = (1-\theta)^r$, $h(x) = \binom{r+x-1}{x}$ for $x = 0, 1, \ldots$, $\pi(\theta) = \log(\theta)$, $t(x) = x$, $\Pi = (-\infty, 0)$.

(c) $$\begin{aligned} f(x|\theta) &= (\sqrt{2\pi}\sigma)^{-1} \exp\{-(x-\mu)^2/(2\sigma^2)\} \\ &= (\sqrt{2\pi}\sigma)^{-1} \exp\{-\frac{x^2}{2\sigma^2} + \frac{x\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\}. \end{aligned}$$

So, $k = 2$, $c(\theta) = (\sqrt{2\pi}\sigma)^{-1}\exp\{-\mu^2/(2\sigma^2)\}$, $h(x) \equiv 1$, $\pi_1(\theta) = -1/(2\sigma^2)$, $\pi_2(\theta) = \mu/\sigma^2$, $t_1(x) = x^2$, $t_2(x) = x$, and $\Pi = \{(\pi_1, \pi_2) : \pi_1 < 0, -\infty < \pi_2 < \infty\}$. For a sample of size n, $(\sum X_i, \sum X_i^2)$ is sufficient for θ, and so is $(\overline{X}_n, s_x^2)$.

(d) $$f(x|\alpha,\beta) = 1/(\Gamma(\alpha)\beta^\alpha) \exp\{-x/\beta + (\alpha-1)\log(x)\} I_{(0,\infty)}(x).$$

So, $k = 2$, $c(\alpha,\beta) = 1/(\Gamma(\alpha)\beta^\alpha)$, $h(x) = x^{-1} I_{(0,\infty)}(x)$, $\pi_1 = -1/\beta$, $t_1(x) = x$, $\pi_2 = \alpha$, $t_2(x) = \log(x)$, $\Pi = \{(\pi_1, \pi_2) : \pi_1 < 0, \pi_2 > 0\}$.

(e) $$f(x|\alpha,\beta) = \Gamma(\alpha+\beta)/(\Gamma(\alpha)\Gamma(\beta)) \exp\{(\alpha-1)\log(x) + (\beta-1)\log(1-x)\} I_{(0,1)}(x).$$

So, $k = 2$, $c(\alpha,\beta) = \Gamma(\alpha+\beta)/(\Gamma(\alpha)\Gamma(\beta))$, $h(x) = x^{-1}(1-x)^{-1} I_{(0,1)}(x)$, $\pi_1 = \alpha$, $t_1(x) = \log(x)$, $\pi_2 = \beta$, $t_2(x) = \log(1-x)$, $\Pi = \{(\pi_1, \pi_2) : \pi_1 > 0, \pi_2 > 0\}$.

3.5.2. From (3.55), $-\log c(\underline{\pi}) = \log \int \exp\{\sum \pi_i t_i(x)\}\, h(x)\, dx$. From Lemma 3, we may pass derivatives beneath the integral sign and obtain

$$\begin{aligned} -\partial \log c(\underline{\pi})/\partial \pi_k &= \int t_k(x) \exp\{\sum \pi_i t_i(x)\}\, h(x)\, dx / c(\underline{\pi})^{-1} \\ &= E_{\underline{\pi}}\{t_k(X)\}, \end{aligned}$$

valid at all interior points. Hence, $E_{\underline{\pi}}\{t_k(X)\}$ exists at all interior points and we can take a second derivative,

$$\begin{aligned} -\partial^2 \log c(\underline{\pi})/\partial \pi_j \partial \pi_k &= \int t_j(x) t_k(x) \exp\{\sum \pi_i t_i(x)\} h(x)\, dx\, c(\underline{\pi}) \\ &\quad + \int t_k \exp\{\sum \pi_i t_i(x)\} h(x)\, dx\, \partial c(\underline{\pi})/\partial \pi_j \\ &= E_{\underline{\pi}}\{t_j(X) t_k(X)\} - E_{\underline{\pi}}\{t_k(X)\} E_{\pi}\{t_j(X)\} \\ &= \mathrm{Cov}_{\underline{\pi}}(t_j(X), t_k(X)). \end{aligned}$$

3.5.3. The joint density of $X_1, \ldots, X_n$ is

$$f_{\mathbf{X}}(\mathbf{x}|\underline{\pi}) = c(\underline{\pi})^n \prod_1^n h(x_i)\ \mathrm{I}(\pi_1 < \min x_i, \max x_i < \pi_2)$$

where $c(\underline{\pi})^{-1} = \int_{\pi_1}^{\pi_2} h(x)\, dx$. Let $T_1 = \min X_i$ and $T_2 = \max X_i$. Then for $\pi_1 < t_1 < t_2 < \pi_2$,

$$\mathrm{P}(T_1 > t_1, T_2 \le t_2 | \underline{\pi}) = \mathrm{P}(\text{all } X_i \in (t, t_2] | \underline{\pi}) = c(\underline{\pi})^n \left(\int_{t_1}^{t_2} h(x)\, dx \right)^n.$$

From this we conclude

$$
\begin{aligned}
f_{T_1,T_2}(t_1,t_2|\underline{\pi}) &= -\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\mathrm{P}(T_1 < t_1, T_2 \geq t_2|\underline{\pi}) \\
&= n(n-1)c(\underline{\pi})^n \left(\int_{t_1}^{t_2} h(x)\,dx\right)^{n-2} h(t_1)h(t_2)\mathrm{I}(\pi_1 < t_1 < t_2 < \pi_2) \\
&= c(\underline{\pi})^n h_0(\mathbf{t})\mathrm{I}_{(\pi_1,\infty)}(t_1)\mathrm{I}_{(-\infty,\pi_2)}(t_2),
\end{aligned}
$$

where

$$
h_0(\mathbf{t}) = n(n-1)\left(\int_{t_1}^{t_2} h(x)\,dx\right)^{n-2} h(t_1)h(t_2)\mathrm{I}(t_1 < t_2).
$$