Large Sample Theory
Ferguson

Exercises, Section 3, Convergence in Law.

1. Prove that $X_n \xrightarrow{\mathcal{L}} X$ if, and only if, $\mathrm{E}g(X_n) \to \mathrm{E}g(X)$ for all bounded differentiable functions g.

2. (a) Show that the characteristic function of the $\mathcal{N}(0,1)$ distribution is $\varphi(t) = e^{-t^2/2}$.
(b) Show that the characteristic function of the $\mathcal{P}(\lambda)$ distribution is $\varphi(t) = e^{-\lambda(1-e^{it})}$.
(c) Show for the characteristic function, $\varphi_X(t)$, of an arbitrary random variable X, that $\varphi_{a+bX}(t) = e^{ita}\varphi_X(tb)$.
(d) Let X be $\mathcal{P}(\lambda)$, and let $Y = (X-\lambda)/\sqrt{\lambda}$. Show that $\varphi_Y(t) \to e^{-t^2/2}$ as $\lambda \to \infty$.
(e) What do you conclude from (d)?

3. For $n = 1, 2, \ldots$, let X_n be a geometric random variable on the non-negative integers, with $P(X_n = 0) = \lambda/n$ for some $\lambda > 0$.
(a) Find the characteristic function of X_n.
(b) Use the method of characteristic functions to show that X_n/n converges in law to an exponential distribution.

4. Let X be a random variable with negative binomial distribution, $P(X = x) = \binom{r+x-1}{x}(1-p)^r p^x$, $x = 0, 1, 2, \ldots$, for some $0 < p < 1$ and $r > 0$. Then X represents the number of successes before the rth failure in a sequence of Bernoulli trials.
(a) Show that the characteristic function of X is $\phi(t) = (1-p)^r/(1-pe^{it})^r$.
(b) Using characteristic functions, find the limiting distributon of X as $r \to \infty$ and $p \to 0$ in such a way that $rp \to \lambda$ for some positive finite number λ.

5. (a) Find the characteristic function of the gamma distribution, $\mathcal{G}(\alpha, \beta)$, whose density is

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-x/\beta} x^{\alpha-1} \qquad \text{for } x > 0.$$

(b) Let X_n have the gamma distribution $\mathcal{G}(n^2, 1/n)$. Using characteristic functions, show that $X_n - n \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$.