Large Sample Theory
Ferguson

Exercises, Section 2, Partial Converses to Theorem 1.

1. Suppose X has the gamma distribution $\mathcal{G}(\alpha, \beta)$ (density proportional to $e^{-x/\beta}x^{\alpha-1}$ on $x > 0$), and let $Y = (1/\gamma)\log(X)$ where $\gamma \neq 0$.

(a) Find the density of Y. Make the change of parameter θ for β through the equation $\log(\beta) = \gamma\theta - \log(\alpha)$, and note that θ is a location parameter of the resulting distribution.

(b) Let $\psi(\alpha)$ denote the digamma function, $\psi(\alpha) = (d/d\alpha)\log(\Gamma(\alpha)) = \Gamma'(\alpha)/\Gamma(\alpha) = (1/\Gamma(\alpha))\int_0^\infty \log(x)e^{-x}x^{\alpha-1}\,dx$. What is the mean of Y? Let σ^2 denote the variance of Y and show that $\sigma^2\gamma^2 = \psi'(\alpha)$. ($\psi'(\alpha)$ is sometimes called the trigamma function.)

(c) Denote the above distribution of Y as $\mathcal{N}(\theta, \sigma^2, \gamma)$, defined for all θ, all $\sigma^2 > 0$ and all $\gamma \neq 0$. Fill in the missing case, $\gamma = 0$, by showing that as $\gamma \to 0$, $\mathcal{N}(\theta, \sigma^2, \gamma)$ converges in law to $\mathcal{N}(\theta, \sigma^2)$, the normal distribution with mean θ and variance σ^2. (You may use $\alpha\psi'(\alpha) \to 1$ as $\alpha \to \infty$, and Stirling's formula.)

(d) Thus $\mathcal{N}(\theta, \sigma^2, \gamma)$ is a three parameter generalization of the normal distribution, with $\mathcal{N}(\theta, \sigma^2, 0)$ being the normal distributions. Show $\mathcal{N}(\theta, \sigma^2, \gamma)$ can also be defined at $\gamma = \pm\infty$, by showing that $\mathcal{N}(0, \sigma^2, \gamma)$ converges in law to the exponential distribution with mean σ as $\gamma \to -\infty$. (You may use $\alpha\Gamma(\alpha) \to 1$ and $\alpha^2\psi'(\alpha) \to 1$ as $\alpha \to 0$.)

2. Someone is walking on the integer lattice of the line starting at some unknown integer, N. Your task is to find him even though you are blindfolded. You may only ask questions of the form "Are you at integer n?" and he must answer truthfully. He may move at most two integers up or down between questions, so after the first question he can only move to one of $N-2, N-1, N, N+1, N+2$. Can you devise a sequence of questions so that you will eventually find him almost surely (i.e. with probability one) no matter where he starts and what he does?

3. Let $X_1, X_2, \ldots$ be a sequence of random variables such that X_1 is uniform on $[0, 1]$, and where for $n = 1, 2, \ldots$, the conditional distribution of X_{n+1} given $X_1, \ldots, X_n$, is uniform on $[0, cX_n]$ for some number c such that $\sqrt{3} < c < 2$.

(a) Find the expectation of X_n^r for $r > 0$.

(b) Show that X_n converges to 0 in mean ($r = 1$), but not in quadratic mean, ($r = 2$).

(c) Does X_n converge to 0 almost surely?

4. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables (with no assumptions of any finite moments). Does X_n/n converge almost surely to 0? If so, show it; if not, give a counterexample.

5. Suppose that random variables $X_1, X_2, \ldots$ are uniformly bounded, i.e. suppose there is a constant B such that $|X_n| < B$ a.s. for all n. Let $r > 0$. Show that $X_n \xrightarrow{\text{P}} X$ if and only if $X_n \xrightarrow{\text{r}} X$.

6. Suppose that X has a standard Cauchy distribution. Find a sequence of random variables X_n for $n = 1, \ldots$, such that X_n converges in quadratic mean to X, but that X_n does not converge to X almost surely.

7. Let $X_1, X_2, \ldots$ be independent Bernoulli trials with $\mathrm{P}(X_n = 1) = 1/n^2$ for $n = 1, 2, \ldots$. Then, as noted in the text, $\mathrm{P}(X_n = 1 \text{ i.o.}) = 0$. Therefore, there is a last n such that $X_n = 1$. Let $N = \max\{n : X_n = 1\}$ be the random time at which this occurs. Find the distribution of N, i.e. find $\mathrm{P}(N = n)$ for $n = 1, 2, \ldots$. What is $\mathrm{E}(N)$?